Testing for changes in the unconditional variance of financial time series

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Abstract: Inclan and Tiao (1994) proposed a test for the detection of changes in unconditional variance which has been used in financial time series analyses. In this article we demonstrate some serious drawbacks to the test when used with this type of data. More specifically, it has big size distortions for leptokurtic and platykurtic innovations. Moreover, the size distortions are more extreme for heteroskedastic conditional variance processes. These results invalidate the test's practical use for financial time series. To overcome these problems, we propose new tests that take the fourth order moment properties of disturbances and conditional heteroskedasticity into explicit account. Monte Carlo experiments demonstrate the better performance of these tests. The new tests' application to the same series in "Aggarwal *et al*. (1999)" reveal that the changes in variance they detect are spurious.

JEL classification: C12, C22, G19

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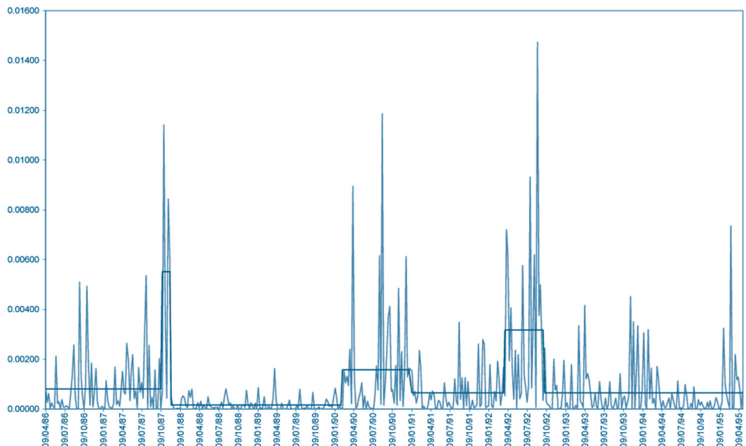
1. INTRODUCTION

Inclan and Tiao (1994), referred to as *IT* hereafter, proposed a statistic to test for changes in the unconditional variance of a stochastic process. This test is based on the assumption that the disturbances are independent and Gaussian distributed, conditions that could be considered unrealistic for financial time series given that they usually show empirical distributions with fat tails (leptokurtic) and persistence in the conditional variance. Despite this, the test has been extensively used for detecting changes in the volatility of financial time series such as returns (see, among others, Wilson et al., 1996, Aggarwal, Inclan and Leal, 1999, and Huang and Yang, 2001). For instance, Figure 1 shows the changes in the unconditional variance variance detected by "Aggarwal *et al*. (1999)" using the *IT* procedure. As the figure shows, several breaks are detected, some of them lasting few observations, which casts doubt on the real number of changes that can be obtained by the application of the *IT* method.

Figure 1
Squared returns from the Nikkei Index and detected changes in the unconditional variance using the Inclan-Tiao test



In this paper we show that the asymptotic distribution of the *IT* test is only free of nuisance parameters when the stochastic process is mesokurtic and the conditional variance is constant. Otherwise, the distribution will depend on certain parameters, and size distortions can be expected in the test when the process is non-mesokurtic and/or there is some persistence in the conditional variance. This will lead to the discovery of spurious changes in the unconditional variance. To overcome these problems, we propose new tests that take into account both the fourth order moment of the process and persistence in the variance. These tests have an asymptotic distribution without nuisance parameters and they belong to the CUSUM-type test family (see Andreou and Ghysels (2002) for a discussion on recent literature). Moreover, we will also show that the *IT* test diverges when the disturbances are IGARCH.

The paper is structured as follows. Section 2 considers the *IT* test in detail, together with its asymptotic distribution for both mesokurtic and non-mesokurtic processes. It then presents a new test that takes the fourth order moment of the process into explicit account. Section 3 focuses on processes where there is persistent conditional variance. The first tests, which do not take this persistence into account, are shown to have asymptotic distributions which depend on nuisance parameters. Subsequently, a modified version of the *IT* is proposed and the asymptotic behavior of the three tests is also considered for IGARCH processes. Section 5 considers the Iterated Cumulative Sum of Squares (ICSS) algorithm suggested by Inclan and Tiao (1994) and adapts it to the suggested new tests. Given that this procedure requires the computation of the test for different sample sizes, we estimate response surfaces to generate critical values for any sample size. In Section 6, a series of Monte Carlo experiments confirm that the limit results obtained in the preceding sections are also relevant in finite samples. The main conclusion derived from these simulations is that the test we propose, which takes into account both persistent variance and the kurtosis of the distribution, outperforms the other two tests and it should therefore be used instead in applied research. In Section 7 we apply the ICSS procedure with the new tests to the same series considered in "Aggarwal *et al.* (1999)" and we show that the changes in variance they detect are spurious. Finally, Section 8 outlines our conclusions. The proofs of all the paper's propositions are shown in the Appendix.

2. THE INCLAN-TIAO TEST

In order to test the null hypothesis of constant unconditional variance, Inclan and Tiao (1994) proposed the use of a statistic given by

$$IT = \sup_k \left| \sqrt{T/2} D_k \right|$$

where

$$D_k = \frac{C_k}{C_T} - \frac{k}{T}$$

and $C_k = \sum_{t=1}^{k} \varepsilon_t^2$, $k = 1,...,T$, is the cumulative sum of squares of ε_t.Under the assumption that ε_t are zero-mean normally, identically and independently distributed random variables, $\varepsilon_t \sim iidN(0,\sigma^2)$, the asymptotic distribution of the test is given by:

$$IT \Rightarrow \sup_r \left| W^*(r) \right| \tag{1}$$

where $W^*(r) \equiv W(r) - rW(1)$ is a Brownian Bridge, $W(r)$ is a standard Brownian motion and $\Rightarrow$ stands for weak convergence of the associated probability measures. If, for a given sample, IT is greater than a specified critical value, then the null hypothesis is rejected.

The most serious drawback to the *IT* test is the fact that its asymptotic distribution is critically dependent on the assumption that the random variables ε_t have a normal, independent and identical distribution. The following proposition establishes the asymptotic distribution of the test for the rather general case $\varepsilon_t \sim iid(0,\sigma^2)$.

Proposition 1. *If* $\varepsilon_t \sim iid(0,\sigma^2)$ *and* $E(\varepsilon_t^4) \equiv \eta_4 < \infty$ *, then* $IT \Rightarrow \sqrt{\frac{\eta_4 - \sigma^4}{2\sigma^4}} \sup_r \left| W^*(r) \right|$.

Hence, the distribution is not free of nuisance parameters and size distortions should be expected when using the critical values of the supremum of a Brownian Bridge. Note that for Gaussian processes, $\eta_4 = 3\sigma^4$ and $IT \Rightarrow \sup_r |W^*(r)|$. When $\eta_4 > 3\sigma^4$, the distribution is leptokurtic (heavily tailed) and too many rejections of the null hypothesis of constant variance should be expected, with an effective size greater than the nominal one. In contrast, when $\eta_4 < 3\sigma^4$ the test will be too conservative. In Section 6 the finite-sample performance of *IT* in such cases will be studied.

Proposition 1 suggests the following correction to the previous test, which will be free of nuisance parameters for identical and independent zero-mean random variables:

$$\kappa_1 = \sup_k \left| T^{-1/2} B_k \right|$$

where

$$B_k = \frac{C_k - \frac{k}{T} C_T}{\sqrt{\hat{\eta}_4 - \hat{\sigma}_4}}$$

$\hat{\eta}_4 = T^{-1} \sum_{t=1}^{T} \varepsilon_t^4$ and $\hat{\sigma}^2 = T^{-1} C_T$. Its asymptotic distribution is established in the following proposition.

Proposition 2. *If* $\varepsilon_t \sim iid(0, \sigma^2)$ *, and* $E(\varepsilon_t^4) \equiv \eta_4 < \infty$ *, then* $\kappa_1 \Rightarrow \sup_r |W^*(r)|$.

Table 1 shows the finite-sample critical values for κ_1. They have been computed from 50,000 replications of $\varepsilon_t \sim iidN(0,1)$, $t = 1, \ldots, T$. A response surface to generate critical values for a wider range of sample sizes will be presented in Section 5.

Table 1
Critical values for κ_1 and κ_2

	κ_1				κ_2			
$\alpha \backslash T$	100	200	500	1000	100	200	500	1000
0.9	1.148	1.167	1.195	1.200	1.170	1.177	1.192	1.197
0.95	1.268	1.300	1.328	1.330	1.269	1.294	1.317	1.329
0.975	1.383	1.420	1.453	1.447	1.352	1.395	1.428	1.442
0.99	1.515	1.547	1.592	1.592	1.448	1.508	1.557	1.586

Note: Computed using 5,000 replications of $\varepsilon_t \sim iidN(0,1)$, $t = 1, \ldots, T$

Given that the asymptotic distribution of this statistic is free of nuisance parameters, we will expect a correct size when the disturbances are *iid*. Section 6 will examine the finite-sample performance for both the *IT* and κ_1 tests. Before that, we consider the case of a conditionally heteroskedastic process.

3. CONDITIONALLY HETEROSKEDASTIC PROCESSES

Both statistics IT and κ_1 in the previous section are reliant on the independence of the sequence of random variables. This is a very strong assumption for financial data, where there is evidence of conditional heteroskedasticity. See, for instance, Bera and Higgins (1993), Bollerslev et al. (1992, 1994) and Taylor (1986). In order to take this specific situation into account, an estimation of the persistence may be used to correct the cumulative sum of squares. Nevertheless, some assumptions regarding ε_t are required.

Assumptions A1: *Assume that the sequence of random variables* $\{\varepsilon_t\}_{t=1}^{\infty}$ *satisfies:*

1) $E(\varepsilon_t)=0$ *and* $E(\varepsilon_t^2)=\sigma^2<\infty$ *for all* $t\geq 1$;
2) $\sup_t E\left(|\varepsilon_t|^{\psi+\varepsilon}\right)<\infty$ *for some* $\psi\geq 4$ *and* $\varepsilon>0$;
3) $\omega_4=\lim_{T\to\infty} E\left(T^{-1}\left(\sum_{t=1}^{T}\left(\varepsilon_t^2-\sigma^2\right)\right)^2\right)<\infty$ *exists, and*
4) $\{\varepsilon_t\}$ *is a-mixing with coefficients* α_j *which satisfy* $\sum_{j=1}^{\infty}\alpha_j^{(1-2/\psi)}<\infty$.

This set of assumptions is similar to that of Herrndorf (1984) and Phillips and Perron (1988) but here we need to impose the existence of moments of order greater than four and a common unconditional variance for all the variables in the sequence, which is the hypothesis we wish to test. Obviously, the existence of the fourth order moment restricts the processes we can deal with. For instance, if ε_t is independent and identically distributed as a *t*-Student with three degrees of freedom, this sequence does not fulfil conditions 2 and 3. Note that the second condition does not impose a common fourth order moment so that some sort of non-stationarity is allowed. ω_4 can be interpreted as the long-run fourth order moment of ε_t or the long-run variance of the zero-mean variable $\xi_t\equiv\varepsilon_t^2-\sigma^2$.[2] Condition 4 controls for the degree of independence of the sequence and shows a trade-off between serial dependence and the existence of high order moments. In our case, by imposing the finiteness of the fourth order moment we allow for a greater degree of serial dependence.

This leads us to propose the following statistic:

$$\kappa_2=\sup_k\left|T^{-1/2}G_k\right|$$

where

$$G_k=\hat{\omega}_4^{-1/2}\left(C_k-\frac{k}{T}C_T\right)$$

and $\hat{\omega}_4$ is a consistent estimator of ω_4. One possibility is to use a non-parametric estimator of ω_4 ,[3]

2 Note that when ε_t is a strictly stationary sequence $\omega_4=2\pi f_\xi(0)$ where $f_\xi(\lambda)$, $-\pi\leq\lambda\leq\pi$, is the spectrum of ξ_t .

3 Another possibility is to use a parametric estimation of the long-run variance of ξ_t based on the Akaike estimator of the spectrum. That is $\hat{\omega}_4=\left(1-\hat{\lambda}(1)\right)^{-2}T^{-1}\sum_{t=1}^{T}\hat{e}_t^2$, where $\hat{\lambda}(1)=\sum_{j=1}^{p}\hat{\lambda}_j$, $\hat{\lambda}_j$ and $\hat{e}_t$ are obtained from the autoregression: $\xi_t=\delta+\sum_{j=1}^{p}\lambda_j\xi_{t-j}+e_t$. When computing the Kokoszka and Leipus (2000) test, Andreou and Ghysels (2002) use the VARHAC estimator proposed by den Haan and Levin (1997) for ω_4.

$$\hat{\omega}_4 = \frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2-\sigma^2\right)^2+\frac{2}{T}\sum_{l=1}^{m}w(l,m)\sum_{t=l+1}^{T}\left(\varepsilon_t^2-\sigma^2\right)\left(\varepsilon_{t-l}^2-\sigma^2\right)$$

where $w(l,m)$ is a lag window, such as the Bartlett window, defined as $w(l,m)=1-l/(m+1)$ or the quadratic spectral. This estimator depends on the selection of the bandwidth m, which can be chosen using an automatic procedure as proposed by Newey-West (1994). The consistency of the estimator $\hat{\omega}_4$ requires that $m\to\infty$ when $T\to\infty$ but $m/T\to 0$. Kokoszka and Leipus (2000) proposed a test that is similar to κ_2, starting out, however, from a different set of assumptions. More specifically, they assume an ARCH(∞) process. As can be seen, our framework is more general than that of Kokoszka and Leipus (2000).

Note that if $\xi_t=\varepsilon_t^2-\sigma^2$ is not correlated, then $\hat{\omega}_4\to E\left(\xi_t^2\right)=\eta_4-\sigma^4$. Note also that the difference between B_k and G_k resides in the fact that the former corrects the cumulative sums for the (square root of the) "short-run" variance of ξ_t, $E\left(\xi_t^2\right)=\eta_4-\sigma^4$, whereas the latter corrects for the (square root of the) "long-run" variance of ξ_t, given by ω_4, and that this last correction accounts for the autocovariance structure of ξ_t. Hence, the difference between statistics B_k and G_k is similar to the one between two *t*-ratios in a linear regression, one computed using the (square root of the) residual variance, t_{sr}, and the other computed with the (square root of the) long-run variance, t_{lr}, using, for instance, Newey-West's (1994) correction. It is well known that the t_{sr} statistic suffers from severe size distortions when there are autocorrelated disturbances, whereas t_{lr} is robust in this case. Thus, we may expect κ_1 and also *IT* to have size problems when ξ_t is autocorrelated, that is, when there is conditional heteroskedasticity.

The limit distribution of the statistics for variance-persistent processes is established in the next proposition.

Proposition 3. *Under assumptions A1,*

a) $IT\Rightarrow\sqrt{\frac{\omega_4}{2\sigma^4}}\sup_r\left|W^*(r)\right|$

b) $\kappa_1\Rightarrow\sqrt{\frac{\omega_4}{\eta_4-\sigma^4}}\sup_r\left|W^*(r)\right|$

c) $\kappa_2\Rightarrow\sup_r\left|W^*(r)\right|$.

Table 1 shows some finite-sample critical values for κ_2 computed from 50,000 replications of $\varepsilon_t\sim iidN(0,1)$, $t=1,\ldots,T$. A response surface to summarize the finite-sample critical values will be presented in Section 5.

For conditionally heteroskedastic processes, one would expect the long-run fourth order moment to be greater than its short-run counterpart $\eta_4-\sigma^4$ and, consequently, that *IT* and κ_1 would have an effective size greater than the nominal one. Let us consider some simple cases. The ARCH(1) process (see Engle, 1982) is given by $\varepsilon_t=u_t\sqrt{h_t}$, where $u_t\sim iidN(0,1)$ and $h_t=\delta+\gamma\varepsilon_{t-1}^2$, conditional on ε_0^2, with $\delta\geq 0$ and $0<\gamma<1$. For $\gamma<3^{-1/2}$, which is the condition for the existence of the fourth order moment, we get:

$$\eta_4=\frac{\delta^2}{(1-\gamma)^2}\frac{3\left(1-\gamma^2\right)}{\left(1-3\gamma^2\right)}$$

and

$$\omega_4 = \frac{2\delta^2}{(1-\gamma)^4(1-3\gamma^2)}.$$

In these circumstances, $\frac{\omega_4}{2\sigma^4} = \frac{1}{(1-\gamma)^2(1-3\gamma^2)} \geq 1$ and the *IT* test will tend to reject the null hypothesis of constant unconditional variance too often. Moreover, given that $\frac{\omega_4}{\eta_4-\sigma^4} = \frac{1}{(1-\gamma)^2} \geq 1$, we should also expect the κ_1 test to suffer an overrejection of the null of constant unconditional variance. In Section 4 these findings are confirmed for finite-samples.

For the GARCH(1,1) processes (see Bollerslev, 1986), the conditional variance is given by:

$$h_t = \delta + \beta h_{t-1} + \gamma\varepsilon_{t-1}^2 \tag{2}$$

The fourth order moment exists if $\beta^2 + 2\beta\gamma + 3\gamma^2 < 1$ and it is given by:

$$\eta_4 = \frac{3\delta^2(1+\gamma+\beta)}{(1-\gamma-\beta)(1-\beta^2-2\beta\gamma-3\gamma^2)}$$

with excess kurtosis:

$$\frac{\eta_4}{\sigma^4} - 3 = \frac{6\gamma^2}{1-\beta^2-2\beta\gamma-3\gamma^2} > 0$$

and long-run fourth order moment:

$$\omega_4 = \frac{2\delta^2(1-2\beta\gamma-\beta^2)(1-\beta)^2}{(1-\gamma-\beta)^4(1-\beta^2-2\beta\gamma-3\gamma^2)}$$

So, if $\beta^2 + 2\beta\gamma + 3\gamma^2 < 1$, which is the condition for the existence of the fourth order moment, we get

$$\frac{\omega_4}{2\sigma^4} = \frac{(1-2\beta\gamma-\beta^2)(1-\beta)^2}{(1-\gamma-\beta)^2(1-\beta^2-2\beta\gamma-3\gamma^2)} > 1$$

and

$$\frac{\omega_4}{\eta_4-\sigma^4} = \frac{(1-\beta)^2}{(1-\gamma-\beta)^2} > 1$$

.

Hence, as in the ARCH(1) case, we can expect the effective size of *IT* and κ_1 to be greater than the nominal one.

Similar results can be expected when dealing with higher order GARCH processes.[4] To sum up, we would expect an overrejection of the null hypothesis for the IT and the κ_1 tests when they are applied to conditionally heteroskedastic processes.

4 The conditions for the existence of fourth order moments in the broad family of GARCH processes, where $h_t^\lambda = g(u_{t-1}) + c(u_{t-1})h_{t-1}^\lambda$, $l > 0$, can be found in Ling and McLeer (2002).

4. A NON-CONSTANT FOURTH ORDER MOMENT

As shown in the previous section, the existence of the fourth order moment, rather than its constancy, and the finiteness of the long-run fourth order moment are both required to establish the asymptotic distribution of the tests. This restricts the class of (G)ARCH processes we can deal with using this theory. Nevertheless, even though the results of Proposition 3 are no longer applicable to all situations, we can try to shed light on some special cases.

Let us consider a simple case, such as the covariance-stationary GARCH(1,1) process given by (2) but with a non-constant fourth order moment. That is, $\beta^2+2\beta\gamma+3\gamma^2 \geq 1$ and $\beta+\gamma<1$. In this case, as shown by Ding and Granger (1996), equation (A.16),

$$E(\varepsilon_t^4)=\eta_{4,t}=3\delta^2\frac{1+\gamma+\beta}{1-\gamma-\beta}\sum_{i=0}^{t}(\beta^2+2\beta\gamma+3\gamma^2)^i$$

tends to infinity. So, the long-run fourth order moment will also be time varying and will tend to infinity. In consequence, according to Proposition 3, we can expect the *IT* test to diverge and it will tend to detect too many changes in variance. Note that this result holds irrespective of whether $T^{-1/2}(C_k-\frac{k}{T}C_T)$, the numerator of the statistic, diverges or not.

Moreover, assuming a distant starting point for the process, the autocorrelation function of will be constant and it will be approximately given by ξ_t^2 $\rho_k \approx (\gamma+\frac{1}{3}\beta)(\gamma+\beta)^{k-1}$, which will decrease exponentially, as shown by Ding and Granger (1996). Thus,

$$\frac{\omega_{4,t}}{E(\varepsilon_t^4)-\sigma^4}=\left(1+2\sum_{j=1}^{\infty}\rho_j\right)$$

$$\approx 1+2\sum_{j=1}^{\infty}\left(\gamma+\frac{1}{3}\beta\right)(\gamma+\beta)^{j-1}$$

$$=\frac{1+\gamma-\frac{1}{3}\beta}{1-\gamma-\beta}>1$$

in such a way that, according to Proposition 3, we can expect an overrejection for the κ_1 test. If $T^{-1/2}(C_k-\frac{k}{T}C_T)$ also diverges, then the distortions in the size of the test will be greater.

For the κ_2 test we may expect the numerator and $\omega_4^{1/2}$ to tend to diverge, so it is difficult to guess how the test will be affected in this case. The Monte Carlo experiments in Section 6 show that the κ_2 is not seriously affected whereas *IT* or κ_1 have dramatic size distortions.

Let us now consider the case of covariance non-stationary processes. We will restrict ourselves to the case of IGARCH(1,1) disturbances, although the generalization to IGARCH(p,q) is straightforward. The following proposition establishes the distribution of the tests for IGARCH disturbances.

Proposition 4. *If* ε_t *is an IGARCH(1,1) process then:*

a) $IT \approx O_p(T^{1/2})$;

b) $\kappa_1 \approx O_p(T^{1/2})$;

c) $\kappa_2 \approx O_p\left((T/m)^{1/2}\right)$.

where m is the bandwidth of the spectral window used to estimate ω_4 .

In consequence, provided that $m/T \to 0$, the tests will diverge, tending to reject the null hypothesis of constant unconditional variance too often. This means that for IGARCH processes, one will find that the tests indicate that the variance is not constant. In this case, the correct procedure is to estimate an IGARCH process rather than trying to model changes in the unconditional variance. The explanation for this expected result is that, like usual unit root tests, the aforementioned test cannot distinguish between I(1) processes and those with structural breaks (see, for instance, Perron,1990). Nevertheless, provided that $(T/m)^{1/2} < T^{1/2}$, we can expect fewer rejections of the null hypothesis when using the κ_2 test.

5. THE ITERATIVE PROCEDURE

The iterative procedure proposed by Inclan and Tiao (1994) for detecting multiple changes in variance, known as the Iterated Cumulative Sum of Squares (ICSS) algorithm, can also be used with the κ_1 and κ_2 tests. A detailed description of the algorithm can be found in Inclan and Tiao (1994). The method entails computing the test several times for different sample sizes. However, using a single critical value for any sample size may distort the performance of the iterative procedure. To overcome this drawback, we fitted response surfaces to the finite-sample critical values of the three tests. Response surfaces are widely used to approximate complex (asymptotic) distributions and to generate finite-sample critical values for statistics that converge to these distributions (see, for instance, MacKinnon, 1994, for more details on this methodology). The idea is to fit a regression of a type such as:

$$q_{i,T}^{\alpha} = \sum_{j=1}^{d} \theta_{i,p_j}^{\alpha} T^{p_j} + v_{i,T} \tag{3}$$

where $q_{i,T}^{\alpha}$ is the quantile a of test $i = \{IT, \kappa_1, \kappa_2\}$ for a sample size *T*; θ_{i,p_j}^{α} , $j = \{1,...,d\}$ are a set of parameters to be estimated and the regressors are powers of the sample size. The values of $q_{i,T}^{\alpha}$ were obtained from Monte Carlo experiments, each of them consisting of 50,000 replications of the process $\varepsilon_t \sim iidN(0,1)$, $t = \{1,...,T\}$ and the corresponding test and the empirical quantiles were computed. The sample sizes considered were *T* = {15, 16, ..., 30, 32, ..., 50, 55, ..., 100, 110, ..., 200, 225, ..., 400, 450, ..., 700, 800, 900, 1,000}. Therefore, 63 experiments for each test were carried out, obtaining 63 independent observations of $q_{i,T}^{\alpha}$ which vary with *T*. Finally, response surfaces as in (3) were fitted to the empirical quantiles. Table 2 shows the final estimates of the response surfaces for a 5% significance level, $\hat{\theta}_{i,p_j}^{0.05}$, as well as some diagnostics.[5] It is worth noting that the fit can be considered quite good ($R^2 > 0.99$ in all cases), and that the residual standard deviations, σ , and the maximum residual are very small.

This table, together with (3), allows us to compute the 5% critical value for a given sample size *T*. For instance, the 5% critical value for the κ_1 statistic for a sample size *T* = 200 can be computed as:

5 The complete set of results for the 1%, 2.5% and 10% significance levels are available from the authors upon request. A GAUSS routine to compute the ICSS algorithm with (any of) the three tests is also available on request. Also, OX routines implemented by Michail Karoglou and based on our GAUSS code are available.

$$\hat{q}^{0.05}_{\kappa_1,200} = 1.363934 - 0.942936 \times 200^{-0.5} + 0.500405 \times 200^{-1} = 1.2998$$

a value very close to the one reported in Table 1.

Table 2
Response surface for the 5% quantiles of the tests

	IT	κ_1	κ_2
$p_1 = 0$	1.359167	1.363934	1.405828
	(771.8)	(846.1)	(75.31)
$p_2 = -0.5$	−0.737020	−0.942936	−3.317278
	(−22.75871)	(−30.64)	(−4.24)
$p_3 = -1$	−0.691556	0.500405	31.22133
	(−6.03)	(4.70)	(3.68)
$p_4 = -2$			−1672.206
			(−5.66)
$p_5 = -3$			52870.53
			(8.92)
$p_6 = -4$			−411015
			(−9.64)
R^2	0.996566	0.995914	0.998772
σ_v	0.003659	0.003492	0.013052
$\max_T \lvert \hat{v}_{i,T} \rvert$	0.01202	0.00847	0.04374

Note: $q^{0.05}_{i,T} = \sum_{j=1}^{d} \theta^{0.05}_{i,p_j} T^{p_j} + v_{i,T}$, where $q^{0.05}_{i,T}$ is the 5%-quantile, based on 50,000 replications of test $i = \{IT,\ \kappa_1,\ \kappa_2\}$ for a sample size T. 63 different sample sizes were considered. White's heteroskedasticity-consistent t-ratios between brackets. For the κ_2 test we have used the quadratic spectral window with automatic bandwidth selection (Newey-West, 1994).

6. MONTE CARLO EXPERIMENTS

In this section we will study the finite-sample performance of the three considered tests as well as the ICSS algorithm. Although the algorithm and the *IT* test have been extensively applied in empirical analyses of financial time series, little attention has been paid to the study of their finite-sample properties. One exception is the study by Andreou and Ghysels (2002). Our simulation experiments complement the aforementioned article. More particularly, we will consider their size for *iid* non-mesokurtic sequences, for ARCH(1) and IGARCH(1,1) processes, and their power when there are some breaks in the unconditional variance. Obviously, applied researchers will be interested in the iterative procedure. Nevertheless, to shed light on the performance of this method when used with the three tests, we begin by analyzing the size and power of the individual tests. All the rejection frequencies were computed using a 5% nominal significance level.

6.1. The size and power of the tests

The first Monte Carlo experiment consisted of generating sequences of *iid* zero-mean random variables with different kurtosis. More specifically, we took into account the uniform distribution $U(-0.5, 0.5)$, and the standard normal, $N(0, 1)$, standard logistic, standard Laplace, standard exponential (with parameter 1) and standard Lognormal distributions. The following table shows the rejection frequencies for the tests.

Table 3
Rejection frequencies for the tests. Non-mesokurtic independent sequences

	Excess kurtosis	$T = 100$			$T = 500$		
		IT	κ_1	κ_2	IT	κ_1	κ_2
Uniform	-1.2	0.0003	0.0570	0.0583	0.0003	0.0500	0.0530
Normal	0	0.0570	0.0567	0.0517	0.0527	0.0503	0.0537
Logistic	1.2	0.1660	0.0497	0.0450	0.1857	0.0473	0.0467
Laplace	3	0.3243	0.0397	0.0423	0.3830	0.0450	0.0470
Exponential	6	0.4597	0.0280	0.0277	0.6360	0.0343	0.0370
Lognormal	≈ 110	0.8130	0.0240	0.0213	0.9700	0.0150	0.0153

Note: Computed using 3,000 replications of $\varepsilon_t \sim iid$, $t = 1, \dots, T$

As can be seen, the *IT* test suffers from severe distortions for non-mesokurtic processes. As predicted from our asymptotic results, it never tends to reject the null hypothesis of constant unconditional variance for platikurtic distributions whereas it tends to reject the null too often for leptokurtic sequences. The two proposed tests are not seriously affected.

The following table shows the rejection frequencies of the three tests when the data generation process is an ARCH(1) process. As expected from our theoretical analysis, all the tests except κ_2 suffer from severe size distortions, as they ignore the persistence in the conditional variance. In contrast, κ_2 seems to have good size properties, even for ARCH processes without a constant fourth order moment (table 4)

The next table shows the rejection frequencies for IGARCH(1,1) processes. Here all three tests tend to reject the null hypothesis of constant variance when the DGP is an IGARCH process. This overrejection is even worse for large samples (say $T = 500$). For large values of γ, say greater than 0.7, the size of κ_2 is not really seriously distorted. For these values, the autocorrelations of ε_t^2, given by $\rho_k \approx \frac{1}{3}(1+2\gamma)(1+2\gamma^2)^{-k/2}$ (see Ding and Granger, 1996), quickly tend to zero. On the other hand, for small values of γ, the persistence of ε_t^2 is high, and κ_2 also shows severe distortions (table 5).

Table 4
Rejection frequencies for the tests. ARCH(1) processes

ARCH(1): $\delta = 0.1$						
	$T = 100$			$T = 500$		
γ	IT	κ_1	κ_2	IT	κ_1	κ_2
0.1	0.083	0.083	0.036	0.105	0.095	0.054
0.3	0.256	0.172	0.039	0.346	0.203	0.040
0.5	0.489	0.296	0.035	0.692	0.338	0.044
0.7	0.643	0.359	0.036	0.902	0.426	0.033
0.9	0.765	0.393	0.024	0.963	0.480	0.022

Note: Computed using 1,000 replications of $\varepsilon_t = u_t\sqrt{h_t}$, where $u_t \sim iid\ N(0, 1)$ and $h_t = \delta + \gamma\varepsilon_{t-1}^2$ and $h_0 = \delta/(1-\gamma)$

Table 5
Rejection frequencies for the tests. IGARCH(1,1) processes

IGARCH(1,1)						
Panel A: $\delta = 0.1$						
	$T = 100$			$T = 500$		
γ	IT	κ_1	κ_2	IT	κ_1	κ_2
0.1	0.696	0.704	0.488	0.983	0.970	0.794
0.3	0.767	0.697	0.205	0.990	0.950	0.372
0.5	0.777	0.620	0.101	0.988	0.875	0.142
0.7	0.812	0.588	0.052	0.987	0.779	0.075
0.9	0.834	0.492	0.044	0.988	0.643	0.025
Panel B: $\delta = 0$						
0.1	0.583	0.614	0.427	0.998	0.998	0.958
0.3	0.979	0.963	0.578	1.000	1.000	0.838
0.5	1.000	0.971	0.336	1.000	0.991	0.378
0.7	1.000	0.933	0.150	1.000	0.939	0.143
0.9	1.000	0.799	0.060	1.000	0.782	0.039

Note: Computed using 1,000 replications of $\varepsilon_t = u_t\sqrt{h_t}$, where $u_t \sim iid\ N(0, 1)$ and $h_t = \delta + \gamma\varepsilon_{t-1}^2 + \beta h_{t-1}$ with $\beta + \gamma = 1$ and

Let us now consider the power of the different tests when there is a change in the unconditional variance of the processes. The process is an *iid* $N(0,1)$ sequence for the first half of the sample and *iid* $N(0,1 + \lambda)$ for the second half. The parameter λ, which can be interpreted as the percentage of change in the unconditional variance, ranges from 0.25 to 1.5. As can be seen from Table 6, κ_2 is the least powerful test, although in no case is this lack of power very extreme.

Table 6
Power of the test when there is a change in the variance

	$T = 100$			$T = 500$		
λ	IT	κ_1	κ_2	IT	κ_1	κ_2
0.25	0.097	0.107	0.091	0.355	0.351	0.343
0.5	0.224	0.225	0.191	0.841	0.826	0.818
0.75	0.425	0.389	0.330	0.982	0.982	0.982
1	0.587	0.535	0.423	0.999	0.999	0.996
1.5	0.824	0.770	0.639	1.000	1.000	1.000

Note: Rejections of the null hypothesis. Computed using 1,000 replications of $\varepsilon_t \sim iidN(0,1)$, $t = 1, \ldots, 0.5T$ and $\varepsilon_t \sim iidN(0,1+\lambda)$ for $t = 0.5T+1, \ldots, T$

6.2. The size and power of the iterative procedure

In this subsection, we will study the performance of the ICSS algorithm when based on one of the three tests. Given that the empirical applications of Section 7 have a sample size of about $T = 500$, this was the one considered. Similar qualitative results were obtained for $T = 100$ which are available upon request.

Table 7
Rejection frequencies for the ICSS procedure. Non-mesokurtic independent sequences

	n_0	n_1	n_2	n_3	n_4	$n_{>4}$
	ICSS(IT)					
Uniform	1.000	0.000	0.000	0.000	0.000	0.000
Normal	0.949	0.047	0.004	0.000	0.000	0.000
Logistic	0.835	0.107	0.047	0.008	0.001	0.002
Laplace	0.604	0.186	0.122	0.058	0.025	0.005
Exponential	0.428	0.161	0.183	0.127	0.065	0.036
Lognormal	0.037	0.091	0.125	0.413	0.197	0.137
	ICSS(κ_1)					
Uniform	0.958	0.036	0.006	0.000	0.000	0.000
Normal	0.946	0.047	0.007	0.000	0.000	0.000
Logistic	0.956	0.041	0.002	0.001	0.000	0.000
Laplace	0.955	0.043	0.002	0.000	0.000	0.000
Exponential	0.972	0.027	0.001	0.000	0.000	0.000
Lognormal	0.988	0.010	0.001	0.001	0.000	0.000
	ICSS(κ_2)					
Uniform	0.958	0.037	0.005	0.000	0.000	0.000
Normal	0.942	0.056	0.002	0.000	0.000	0.000
Logistic	0.953	0.044	0.003	0.000	0.000	0.000
Laplace	0.949	0.049	0.002	0.000	0.000	0.000
Exponential	0.968	0.030	0.002	0.000	0.000	0.000
Lognormal	0.985	0.014	0.001	0.000	0.000	0.000

Note: ICSS(i), $i = \{IT, \kappa_1, \kappa_2\}$, stands for the ICSS algorithm based on the i test; n_j, $j = \{0, 1, \ldots, 4, >4\}$ stands for the relative frequency of detecting j changes in variance. $T = 500$.

As in the preceding subsection, we will begin by considering non-mesokurtic independent random sequences. Table 7 shows the frequency of detected changes in the variances when the ICSS procedure is used with the three tests. The more kurtosis the process has, the greater the number of time breaks erroneously detected by the iterative procedure with the *IT* test. In contrast, few of them are found with κ_1 or κ_2.

For conditional variance heteroskedastic sequences, the picture is similar to that of the individual tests: the iterative method based on *IT* or κ_1 tends to discover too many changes in variance, as can be seen in Table 8. The procedure based on κ_2 has a good performance and hardly ever detects any spurious time break. For IGARCH processes, as can be seen in Table 9, this last procedure also outperforms the other two, finding few spurious changes in variance except for small values of γ.

Finally, we studied the power of the ICSS procedure when there are two changes in the unconditional variance of an independent gaussian sequence. The sample size is $T = 500$ and the changes in the variance are located at $T_1 = 200$ and $T_2 = 400$. Two Data Generation Processes (DGP) were considered. DGP 1 is given by $e_t \sim iidN(0,1)$ for $t = 1, \ldots, T_1$ and $t = T_2 + 1, \ldots, T$

Table 8
Rejection frequencies for the ICSS procedure. ARCH(1) processes

γ	n_0	n_1	n_2	n_3	n_4	$n_{>4}$
			ICSS(IT)			
0.1	0.902	0.074	0.021	0.003	0.000	0.000
0.3	0.665	0.138	0.127	0.032	0.026	0.012
0.5	0.317	0.112	0.185	0.132	0.119	0.135
0.7	0.144	0.073	0.094	0.131	0.132	0.426
0.9	0.038	0.030	0.048	0.096	0.091	0.697
			ICSS(κ_1)			
0.1	0.904	0.073	0.021	0.002	0.000	0.000
0.3	0.789	0.128	0.063	0.014	0.004	0.002
0.5	0.677	0.154	0.104	0.042	0.017	0.006
0.7	0.583	0.148	0.129	0.065	0.047	0.028
0.9	0.464	0.145	0.197	0.078	0.066	0.050
			ICSS(κ_2)			
0.1	0.952	0.039	0.009	0.000	0.000	0.000
0.3	0.944	0.050	0.005	0.001	0.000	0.000
0.5	0.969	0.030	0.001	0.000	0.000	0.000
0.7	0.976	0.024	0.000	0.000	0.000	0.000
0.9	0.972	0.025	0.003	0.000	0.000	0.000

Note: See Table 7.

Table 9
Rejection frequencies for the ICSS procedure. IGARCH(1,1) processes

γ	n_0	n_1	n_2	n_3	n_4	$n_{>4}$
			ICSS$\left(IT_T\right)$			
0.1	0.022	0.085	0.159	0.239	0.235	0.260
0.3	0.014	0.010	0.039	0.065	0.095	0.777
0.5	0.010	0.006	0.027	0.041	0.060	0.856
0.7	0.012	0.010	0.033	0.058	0.057	0.830
0.9	0.013	0.013	0.032	0.054	0.064	0.824
			ICSS$\left(\kappa_1\right)$			
0.1	0.035	0.103	0.148	0.202	0.229	0.283
0.3	0.050	0.053	0.081	0.104	0.152	0.560
0.5	0.129	0.066	0.128	0.120	0.116	0.441
0.7	0.230	0.126	0.132	0.121	0.130	0.261
0.9	0.371	0.117	0.199	0.103	0.099	0.111
			ICSS$\left(\kappa_2\right)$			
0.1	0.229	0.271	0.219	0.159	0.088	0.034
0.3	0.625	0.206	0.119	0.042	0.007	0.001
0.5	0.858	0.100	0.035	0.006	0.001	0.000
0.7	0.925	0.062	0.013	0.000	0.000	0.000
0.9	0.964	0.035	0.001	0.000	0.000	0.000

Note: See Table 7.

and $e_t \sim iidN(0,1+\lambda)$ for $t = T_1 + 1, \dots, T_2$. DGP 2 also takes into consideration a zero-mean independent gaussian sequence with variance 1 for $t = 1, \dots, T_1$, $(1 + \lambda)$ for $t = T_1 + 1, \dots, T_2$ and $(1 + \lambda)^{-1}$ for $T_2 + 1, \dots, T$. Table 10 reports the average number of breaks detected with the ICSS algorithm for these two DGP and different values of λ. The procedure based on κ_2 is slightly less powerful than the other two, although the difference is not important.

Thus, we may conclude that the procedures based on IT or κ_1 show large size distortions, invalidating their practical use with financial time series, which are leptokurtic and show persistence in the conditional variance. The procedure based on κ_2 is not affected by these distortions and achieves a similar power profile (see Table 10).

7. EMPIRICAL APPLICATION

In this section we check for the constancy of the unconditional variance of the four financial time series that have already been studied in Aggarwal et al. (1999), where several changes in variance were detected for these series. The data consists of closing values for the stock indexes S&P500 (USA), Nikkei Average (Japan), FT100 (UK) and Hang-Seng (Hong-Kong). The period covers May 1985 to April 1995. We have calculated the weekly returns for Wednesdays.

Table 10
Power of the ICSS procedure when there is a change in the variance

	DGP 1			DGP 2		
λ	IT	κ_1	κ_2	IT	κ_1	κ_2
0.25	0.173	0.171	0.134	0.222	0.213	0.154
0.5	0.691	0.631	0.511	1.382	1.175	0.688
0.75	1.399	1.314	1.061	2.026	1.975	1.312
1	1.860	1.794	1.534	2.112	2.125	1.715
1.5	2.115	2.094	1.973	2.161	2.164	1.864

Note: Average number of breaks detected. Computed using 1,000 replications of DGP 1: $\varepsilon_t \sim iidN(0,1)$ for $t = 1, \ldots, 200$, $\varepsilon_t \sim iidN(0,1+\lambda)$ for $t = 201, \ldots, 400$, and $\varepsilon_t \sim iidN(0,1)$ for $t = 401, \ldots, 500$; DGP 2: $\varepsilon_t \sim iidN(0,1)$ for $t = 1, \ldots, 200$, $\varepsilon_t \sim iidN(0,1+\lambda)$ for $t = 201, \ldots, 400$, and $\varepsilon_t \sim iidN(0,(1+\lambda)^{-1})$ for $t = 401, \ldots, 500$;

When there was no trading on a given Wednesday, the trading day before Wednesday was used to compute the return.

Table 11 presents the descriptive statistics for each of the aforementioned series. All the series show excess kurtosis. The Ljung-Box statistic on the squared series and Engle's Lagrange multiplier test (Engle,1982) for the existence of ARCH effects provide strong evidence of non-constant conditional variance for the four series. So, as concluded from the asymptotic theory as well as the Monte Carlo experiments, we may expect too many rejections using the Inclan-Tiao test.

Table 11
Descriptive statistics

	FT100	Nikkei	S&P	Hang-Seng
Mean	0.00135	0.000169	0.002026	0.003259
Min	-0.17817	-0.10892	-0.16663	-0.34969
Max	0.09822	0.12139	0.06505	0.11046
std. dev.	0.02275	0.02940	0.02084	0.03765
Skewness	-1.54899	-0.51655	-1.45512	-2.31416
Kurtosis	15.8642	4.78076	12.3227	19.6888
$Q2(15)$	88.278	130.93	87.015	38.06
	(0.00)	(0.00)	(0.00)	(0.00)
$LM(2)$	103.69	34.29	65.09	32.577
	(0.00)	(0.009)	(0.00)	(0.00)
$LM(5)$	106.77	62.66	65.51	34.02
	(0.00)	(0.00)	(0.00)	(0.00)

Note: $Q2(15)$ stands for the Ljung-Box statistic on the squared returns for 15 lags and $LM(j)$ for Engle's Lagrange multiplier test for ARCH(j) effects. p-values between brackets.

Table 12 presents the results obtained from using the ICSS algorithm with these series. The second column gives the points of structural changes in variance obtained by Aggarwal et. al. (1999), whilst the remaining columns show the points when the iterative procedure is implemented using the response surfaces shown in Section 4. Comparing the four detected sets of time breaks, several conclusions can be reached. First, comparing the second and third columns, fewer changes in variance are detected when the critical values are adapted to the effective sample size. Second, controlling for the kurtosis of the series (column four) reduces the number of time breaks. Finally, applying the ICSS (κ_2) procedure, which corrects for conditional heteroskedasticity, no changes are observed. From our theoretical results, the Monte Carlo experiments and the descriptive analysis, we can conclude that the detected changes obtained by Aggarwal et al. (1999) and those obtained with the ICSS (IT) method are spurious and that these results might be attributable to the kurtosis and conditional heteroskedasticity of the series.

Table 12
Detected changes in variance with the ICSS algorithm

	AIL	ICSS(IT)	ICSS(κ_1)	ICSS(κ_2)
FT100	14-10-87 (80)			
	23-12-87 (90)			
Nikkei	17-6-87 (63)	14-10-87 (80)	14-10-87 (80)	
	18-11-87 (85)	25-11-87(86)	25-11-87(86)	
	14-2-90 (199)	14-2-90(199)	14-2-90(199)	
	23-01-91 (247)	9-1-91(245)	9-1-91(245)	
	25-3-92 (307)	25-3-92(307)	25-3-92(307)	
	30-9-92 (334)	30-9-92(334)	30-9-92(334)	
S&P	21-5-86(55)	21-5-86(55)	21-5-86(55)	
	7-10-87(127)	7-10-87(127)		
	4-11-87 (131)	4-11-87 (131)		
	10-8-88(171)		1-06-88(161)	
	1-8-90(274)		1-8-90(274)	
	13-2-91(302)		13-2-91(302)	
	22-4-92(364)	15-4-92 (363)	22-4-92(364)	
Hang-Seng	14-10-87(128)	14-10-87(128)		
	4-11-87(131)	4-11-87(131)		
	2-3-88(148)	17-2-88(146)		
	17-5-89(211)	17-5-89(211)		
	12-7-89(219)	12-7-89(219)		
	7-10-92(388)	7-10-92(388)		

Note: Dates of the detected changes in variance (position of the observation between brackets). AIL stands for the results of Aggarwal et al. (1999). ICSS(i), $i = \{IT, \kappa_1, \kappa_2\}$ stands for the ICSS algorithm based on the i test.

8. CONCLUSIONS

In this article we have proven that the test used as a basis for the implementation of Inclan and Tiao's (1994) ICSS algorithm has two serious drawbacks that invalidate its use for financial time series. First, it neglects the fourth order moment properties of the process and, second, it does not allow for conditional heteroskedasticity. The κ_2 test we propose in this paper takes these two features into explicit consideration. Monte Carlo experiments detected extreme size distortions for the *IT* test whereas κ_2 is correctly sized in almost all the considered scenarios and it turns out to be only slightly less powerful.

These theoretical findings lead us to recommend the use of the ICSS procedure implemented with κ_2 and to be skeptical about the results obtained with the method based on the *IT* test. As an example of this, we applied the ICSS method using the three tests considered in this paper to four of the financial time series analyzed in Aggarwal et al. (1999). These authors detected several time breaks in their financial data. The descriptive statistics show that these series are leptokurtic and conditionally heteroskedastic, the two situations where the *IT* test does not work properly. The ICSS procedure, computed using the suggested κ_2 test, does not detect any change in the unconditional variance. Hence, given our findings, the time breaks detected by Aggarwal *et al*. (1999) are spurious.

APPENDIX: PROOF OF THE PROPOSITIONS

We shall make use of the following asymptotic result:

Lemma 1. *Let* $\xi_t \equiv \varepsilon_t^2 - \sigma^2$ be a sequence of random variables that satisfies assumptions A1. Define $r \in [0,1]$. Then, for $T^{-1/2}\omega^{-1/2}\sum_{t=1}^{[rT]}\xi_t \Rightarrow W(r)$, a standard Brownian motion.

Proof. First, note that if $\{\varepsilon_t\}$ is α-mixing, then it is also ξ_t. Next, set of assumptions A1 is a restricted case of the conditions of Herrndorf's Theorem and, hence, the limit distribution stated in the previous lemma follows on directly from that theorem.

Note that the assumptions regarding ε in Propositions 1 and 2 fulfil set of assumptions A1.

Proof Propositions 1 and 2. This proof follows most of the steps used by Inclan-Tiao, so we will only give a brief outline. First, note that $V(\xi_t) = E(\varepsilon_t^2 - \sigma^2)^2 = \eta_4 - \sigma^4 = \omega$ where $\eta_4 \equiv E(\varepsilon_t^4)$, but only for mesokurtic random variables $V(\xi_t) = 2\sigma^4$. Moreover, $T^{-1}C_T = T^{-1}\sum_{t=1}^{T}\varepsilon_t^2 \to \sigma^2$, where $\to$ stands for convergence in probability, and

$$
\begin{aligned}
T^{-1/2}\omega^{-1/2}\left(C_k - \frac{k}{T}C_T\right) &= T^{-1/2}\omega^{-1/2}\left(\sum_{t=1}^{k}\varepsilon_t^2 - \frac{k}{T}\sum_{t=1}^{T}\varepsilon_t^2\right) \\
&= T^{-1/2}\omega^{-1/2}\left(\sum_{t=1}^{k}(\varepsilon_t^2 - \sigma^2) - \frac{k}{T}\sum_{t=1}^{T}(\varepsilon_t^2 - \sigma^2)\right) \\
&= T^{-1/2}\omega^{-1/2}\left(\sum_{t=1}^{k}\xi_t - \frac{k}{T}\sum_{t=1}^{T}\xi_t\right) \\
&\Rightarrow W(r) - rW(1) \equiv W^*(r)
\end{aligned}
$$

where $r \equiv \frac{k}{T} \in [0,1]$. Thus, $T^{-1/2}\left(C_k - \frac{k}{T}C_T\right) \Rightarrow \sqrt{\omega}W^*(r)$,

$$\sqrt{T/2}D_k = \sqrt{T/2}\left(\frac{C_k}{C_T} - \frac{k}{T}\right) \Rightarrow \sqrt{\frac{\omega}{2\sigma^4}}W^*(r),$$

and, applying the Continuous Mapping Theorem (CMT), Proposition 1 is proven. Proposition 2 follows on immediately from the previous one.

Proof Proposition 3. In this situation, the ξ_i are no longer independent. Thus,

$$T^{-1/2}\omega_4^{-1/2}\left(C_k - \frac{k}{T}C_T\right) = T^{-1/2}\omega_4^{-1/2}\left(\sum_{t=1}^{k}\xi_t - \frac{k}{T}\sum_{t=1}^{T}\xi_t\right) \Rightarrow W^*(r).$$

Provided that $\hat{\omega}_4$ is a consistent estimator, $T^{-1/2}\hat{\omega}_4^{-1/2}\left(C_k - \frac{k}{T}C_T\right) = T^{-1/2}G_k \Rightarrow W^*(r)$ and, applying the CMT, result c) is proven. Given that

$$T^{-1/2}\left(C_k - \frac{k}{T}C_T\right) \Rightarrow \omega_4^{1/2}W^*(r),$$

it follows that $\sqrt{T/2}D_k = \sqrt{T/2}\left(\frac{C_k}{C_T} - \frac{k}{T}\right) \Rightarrow \sqrt{\frac{\omega_4}{2\sigma^4}}W^*(r)$ and

$$T^{-1/2}B_k = T^{-1/2}\frac{C_k - \frac{k}{T}C_T}{\sqrt{\hat{\eta}_4 - \hat{\sigma}^4}} \Rightarrow \sqrt{\frac{\omega_4}{\eta_4 - \sigma^4}}W^*(r).$$

Hence, applying the CMT, a) and b) are proven.

We will consider the simplest case of IGARCH(1,1) processes, although the generalization to any IGARCH(p,q) is straightforward. The following lemma provides some intermediate results needed to prove Proposition 4.

Lemma 2. *Let* $\varepsilon_t = u_t\sqrt{h_t}$, *where* $u_t \sim iidN(0,1)$ *and* $h_t = \delta + \beta h_{t-1} + \gamma\varepsilon_{t-1}^2$ *with* $\beta + \gamma = 1$, , $\delta > 0$, $0 \le \beta < 1$ *and* $0 < \gamma < 1$, *conditional on* h_0 *and* ε_0^2. *Assume also that* $E\left[\ln\left(\beta + \gamma\varepsilon_t^2\right)\right] < 0$ *and* $0 < p < \psi/2$ *for* $0 < p < \psi/2$ *and* $\lambda > 0$ *, which ensures the existence of the fourth order moment –see Nelson (1990) Theorem 4. Denote the long-run variance of* $v_t \equiv w_t - \beta w_{t-1}$ *as* $\omega_v = \lim_{T\to\infty} E\left(T^{-1}\left(\sum_{t=1}^{T} v_t\right)^2\right) < \infty$, *where* $w_t \equiv \varepsilon_t^2 - h_t$. *Define* $r \equiv \frac{k}{T} \in [0,1]$. *Then:*

L1) $T^{-2}C_k \to \frac{\delta}{2}r^2$;

L2) $T^{-3}\sum_{t=1}^{T}\varepsilon_t^4 = T^{-2}\hat{\eta}_4 \to \dfrac{\delta^2}{3}$.

Proof. We can write: $\varepsilon_t^2 = \delta + (\beta + \gamma)\varepsilon_{t-1}^2 + w_t - \beta w_{t-1} = \delta + \varepsilon_{t-1}^2 + v_t$. So, v_t is an invertible MA(1) process. Recursive substitution gives: $\varepsilon_t^2 = \varepsilon_0^2 + \delta t + S_t$, where $S_t = \sum_{j=1}^{t} v_j$. Moreover, it is well-known that $\omega_v^{-1/2}T^{-1/2}S_{[rT]} \Rightarrow W(r)$, $r \in [0,1]$.

Let us now consider the cumulative sum of squares:

$$
\begin{aligned}
T^{-2}C_k &= T^{-2}\sum_{t=1}^{k}\varepsilon_t^2 = T^{-2}\sum_{t=1}^{k}\left(\varepsilon_0^2+\delta t+S_t\right)\\
&= \frac{k}{T^2}\varepsilon_0^2+\frac{1}{T^2}\frac{\delta}{2}k(k+1)+T^{-2}\sum_{t=1}^{k}S_t\\
&= \frac{\delta}{2}\left(\frac{k^2}{T^2}+\frac{k}{T^2}\right)+o_p(1)\\
&\to \frac{\delta}{2}r^2
\end{aligned}
$$

$r\equiv\frac{k}{T}\in[0,1]$, when $T\to\infty$, provided that $\sum_{t=1}^{k}S_t\approx O_p\left(k^{3/2}\right)$. Then L1 is proven.

For result L2 note that, $\varepsilon_t^4=\left(\varepsilon_0^2+\delta t+S_t\right)^2=\varepsilon_0^4+\delta^2t^2+S_t^2+2\varepsilon_0^2\delta t+2\varepsilon_0^2S_t+2\delta tS_t$. Then,

$$
\begin{aligned}
T^{-3}\sum_{t=1}^{t}\varepsilon_t^4 &= T^{-3}\left(T\varepsilon_0^4+\delta^2\left(\frac{1}{3}T^3+\frac{1}{2}T^2+\frac{1}{6}T\right)+\sum_{t=1}^{t}S_t^2\right.\\
&\qquad\left.+\varepsilon_0^2\delta\left(T^2+T\right)+2\varepsilon_0^2\sum_{t=1}^{T}S_t+2\delta\sum_{t=1}^{T}tS_t\right)\\
&= \frac{\delta^2}{3}+o_p(1)\\
&\to \frac{\delta^2}{3}
\end{aligned}
$$

provided that $\sum_{t=1}^{T}S_t^2\approx O_p\left(T^2\right)$, $\sum_{t=1}^{T}tS_t\approx O_p\left(T^{5/2}\right)$ and $T^{-3}\varepsilon_0^2\sum_{t=1}^{T}S_t\approx o_p(1)$. Hence, $T^{-2}\eta_4\to\frac{\delta^2}{3}$ and L2 is proven.

Proof Proposition 4. From L1 it follows that:

$$
\begin{aligned}
D_k &= \frac{T^{-2}C_k}{T^{-2}C_T}-\frac{k}{T}\\
&\to r^2-r.
\end{aligned}
$$

Thus, $\sqrt{T/2}D_k\approx O_p\left(T^{1/2}\right)$ and it diverges. Hence, result a) is proven. For result b) we have $T^{-2}\sigma^4\to\frac{\delta^2}{2^2}$, from L1, and using L2:

$$
T^{-1}B_k=\frac{T^{-2}\left(C_k-\frac{k}{T}C_T\right)}{T^{-1}\sqrt{\eta_4-\sigma^4}}\to\frac{\frac{\delta}{2}\left(r^2-r\right)}{\sqrt{\frac{\delta^2}{3}-\frac{\delta^2}{2^2}}}=r(r-1)\sqrt{3}
$$

and then $T^{-1/2}B_k\approx\ O_p\left(T^{1/2}\right)$, so that it diverges. For result c) we have

$$
\begin{aligned}
\varpi_4 &= \frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2-T^{-1}C_T\right)^2+\frac{2}{T}\sum_{l=1}^{m}w(l,m)\sum_{t=l+1}^{T}\left(\varepsilon_t^2-T^{-1}C_T\right)\left(\varepsilon_{t-l}^2-T^{-1}C_T\right)\\
&= T^{-1}\sum_{t=1}^{T}\varepsilon_t^4-T^{-2}C_T^2+\\
&\quad 2\sum_{l=1}^{m}w(l,m)\left(T^{-1}\sum_{t=l+1}^{T}\varepsilon_t^2\varepsilon_{t-l}^2-T^{-2}C_T\sum_{t=l+1}^{T}\varepsilon_t^2-T^{-2}C_T\sum_{t=l+1}^{T}\varepsilon_{t-l}^2+T^{-2}C_T^2\right)
\end{aligned}
$$

Hence, $T^{-4}C_k^2 \to \left(\frac{\delta}{2}r^2\right)^2$, $\quad T^{-3}\sum_{t=1}^{T}\varepsilon_t^4 \to \frac{\delta^2}{3}$

$$\begin{aligned}
T^{-3}\sum_{t=l+1}^{T}\varepsilon_t^2\varepsilon_{t-l}^2 &= T^{-3}\sum_{t=l+1}^{T}\left(\varepsilon_0^2+\delta t+S_t\right)\left(\varepsilon_0^2+\delta(t-l)+S_{t-l}\right) \\
&= T^{-2}\varepsilon_0^4 + T^{-3}\delta\varepsilon_0^2\left(\frac{1}{2}T^2+\frac{1}{2}T-\frac{1}{2}l^2-\frac{1}{2}l\right)+ \\
&\quad T^{-3}\varepsilon_0^2\sum_{t=l+1}^{T}S_t + T^{-3}\delta\varepsilon_0^2\left(\frac{1}{2}T^2+\frac{1}{2}T-lT-\frac{1}{2}l+\frac{1}{2}l^2\right)+ \\
&\quad \delta^2\left(-\frac{1}{6}lT^{-3}+\frac{1}{3}+\frac{1}{2}T^{-1}+\frac{1}{6}T^{-2}+\frac{1}{6}l^3T^{-3}-\frac{1}{2}lT^{-1}-\frac{1}{2}lT^{-2}\right)+ \\
&\quad T^{-3}\delta\sum_{t=l+1}^{T}(t-l)S_t + T^{-3}\varepsilon_0^2\sum_{t=l+1}^{T}S_{t-l} + T^{-3}\delta\sum_{t=l+1}^{T}tS_{t-l} + \\
&\quad T^{-3}\sum_{t=l+1}^{T}S_tS_{t-l} \\
&= \frac{\delta^2}{3}+o_p(1)
\end{aligned}$$

and $T^{-2}\sum_{t=l+1}^{T}\varepsilon_t^2 = \left(\frac{T-l}{T}\right)^2(T-l)^{-2}\sum_{t=l+1}^{T}\varepsilon_t^2 \to \frac{\delta}{2}$, provided that $l/T \to 0$, so that $T^{-4}C_T\sum_{t=l+1}^{T}\varepsilon_t^2 - T^{-4}C_T\sum_{t=l+1}^{T}\varepsilon_{t-l}^2 + T^{-4}C_T^2 \to \left(\frac{\delta}{2}\right)^2$. Thus, for the Bartlett window, $w(l,m)=1-l/(m+1)$ and

$$\begin{aligned}
T^{-2}\varpi_4 &= \left(\frac{\delta^2}{3}-\left(\frac{\delta}{2}\right)^2\right)+2\sum_{l=1}^{m}w(l,m)\left(\frac{\delta^2}{3}-\left(\frac{\delta}{2}\right)^2\right)+o_p(1) \\
&= \sum_{l=-m}^{m}w(l,m)\left(\frac{\delta^2}{3}-\left(\frac{\delta}{2}\right)^2\right)+o_p(1) \\
&\to \frac{1}{12}(2m+1)\delta^2
\end{aligned}$$

so that

$$\begin{aligned}
m^{1/2}T^{-1}G_k &= \left(m^{-1}T^{-2}\varpi_4\right)^{-1/2}\left(T^{-2}C_k-\frac{k}{T}T^{-2}C_T\right) \\
&\to r(r-1)\sqrt{3/2}.
\end{aligned}$$

Then, $T^{-1/2}G_k \approx O_p\left((T/m)^{1/2}\right)$. This completes the proof.

REFERENCES

— Aggarwal R., C. Inclan and R. Leal, 1999. Volatility in Emerging Stock Markets, Journal of Financial & Quantitative Analysis, 34, 33-55.
— Andreou, E. and E. Ghysels, 2002. Detecting Multiple Breaks in Financial Market Volatility Dynamics, Journal of Applied Econometrics, 17, 579-600.
— Bera A. K. and M. L. Higgins, 1993. ARCH Models: Properties, Estimation and Testing, Journal of Economic Surveys, 7, 305-366.
— Bollerslev, T., R. Y. Chou and K. F. Kroner, 1992. ARCH Modeling in Finance. A Review of the Theory and Empirical Evidence, Journal of Econometrics, 52, 5-59.
— Bollerslev, T., R. F. Engle and D. B. Nelson, 1994. ARCH Models. In Handbook of Econometrics, Volume IV, Chapter 49, 2961-3031. Edited by R.F. Engle and D. L. McFadden. Elsevier Science.
— Bollerslev, T., 1986. Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics, 31, 307-27.
— den Haan W.J. and A. Levin, 1997. A Practitioner's Guide to Robust Covariance Matrix Estimation. In Handbook of Statistics, Vol 15. Rao, C.R. and G.S. Maddala, eds). 291-341.
— Ding, Z. and C.W.J. Granger, 1996. Modelling Volatility Persistence of Speculative Returns: A New Approach, Journal of Econometrics, 73, 185-216.
— Engle, R.F., 1982. Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, Econometrica, 50, 987-1008.
— Huang and Yang, 2001. The Impact of Settlement Time on the Volatility of Stock Market Revisited. An Application of the Iterated Cumulative Sums of Squares Detection Method for Changes of Variance, Applied Economic Letters, 8, 665-668.
— Herrndorf, N., 1984. A Functional Central Limit Theorem for Weakly Dependent Sequences of Random Variables, Annals of Probability, 12, 141-153.
— Inclan, C. and G.C. Tiao, 1994. Use of Cumulative Sums of Squares for Retrospective Detection of Changes of Variance, Journal of the American Statistical Association, 89, 913-923.
— Kokoszka, P. and R. Leipus, 2000. Change-Point Estimation in ARCH Models, Bernoulli, 6, 1-28.
— Ling, S. and M. McLeer, 2002. Stationarity and the Existence of Moments of a Family of GARCH Processes, Journal of Econometrics, 109, 109-117.
— MacKinnon, J., 1994.Approximate Asymptotic Distribution Functions for Unit-Root and Cointegration Tests. Journal of Business & Economic Statistics, 12, 167-176.
— Milhøj, A., 1985. The Moment Structure of ARCH Models, Scandinavian Journal of Statistics, 12, 281-92.
— Nelson, D.B., 1990. Stationarity and Persistence in the GARCH(1,1) Model, Econometric Theory, 6, 318-334.
— Newey, W.K. and K.D. West, 1994. Automatic Lag Selection in Covariance Matrix Estimation, Review of Economic Studies, 61, 631-653.
— Perron, P., 1990. Testing for a Unit Root in a Time Series with a Changing Mean, Journal of Business & Economic Statistics, 8, 153-162.
— Phillips, P.C.B. and P. Perron, 1988. Testing for a Unit Root in Time Series Regression, Biometrica, 75, 335-346.
— Taylor, S., 1986. Modeling Financial Time Series. Wiley. NY.
— Wilson B., R. Aggarwal and C. Inclan, 1996. Detecting Volatility Changes across the Oil Sector, The Journal of Futures Markets, 16, 313-330.