

The First Fundamental Theorem of Asset Pricing

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ABSTRACT

This article is a survey on the existing versions of the First Fundamental Theorem of Asset Pricing (FTAP). FTAP provides characterization of the existence of martingale measures in different settings. In the simplest case FTAP establishes the equivalence between the absence of arbitrage and the existence of equivalent martingale measures. However, in a more general setting, for example, with infinite time, infinite number of states of the nature, infinite number of assets or in the presence of market frictions a simple version of FTAP cannot be proved since the absence of arbitrage is not sufficient to construct martingale measures. Accordingly, the literature constructs martingale measures by generalizing the concept of arbitrage to free lunch, free lunch with bounded risk or free lunch with vanishing risk. Following this approach FTAP for markets with frictions is established through the martingale property of a price process lying within the bid and the ask price processes. Finally, recent studies apply the theory of projective systems of probability measures and without relying on free lunches characterize the absence of arbitrage by the existence of projectively equivalent risk-neutral probability measures.

Keywords: Fundamental Theorem of Asset Pricing; Risk-neutral Pricing; Martingale Measures.

JEL Classification: G12, G13.

El primer teorema fundamental de la valoración de activos

RESUMEN

Este artículo es una revisión de las versiones existentes del Primer Teorema Fundamental de la Valoración de Activos (FTAP). El FTAP proporciona una caracterización de la existencia de medidas de martingalas en distintos escenarios. En el caso más sencillo el FTAP establece la equivalencia entre la ausencia de arbitraje y la existencia de medidas equivalentes en martingalas. Sin embargo, en un contexto más general, por ejemplo con tiempo infinito, un infinito número de estados de la naturaleza, un infinito número de activos o en presencia de fricciones de mercado, una versión sencilla del FTAP no puede ser demostrada puesto que la ausencia de arbitraje no es suficiente para constituir las medidas en martingalas. Por tanto, en la literatura se constituyen las medidas en martingalas mediante la generalización del concepto de arbitraje al de FL (*free lunch*), FL con riesgo acotado o FL con riesgo evanescente. Siguiendo este enfoque el FTAP para mercado con fricciones es establecido mediante la propiedad de martingalas de un proceso de precios entre los procesos de precios de compra y venta. Finalmente, estudios recientes aplican la teoría de sistemas proyectivos de medidas de probabilidad y sin depender de FL caracterizan la ausencia de arbitraje mediante la existencia de medidas de probabilidad de riesgos neutrales proyectivamente equivalentes.

Palabras Clave: Teorema Fundamental de la Valoración de Activos, Valoración Riesgo Neutral, Medidas en Martingalas.

Clasificación JEL: G12, G13.

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1. INTRODUCTION

One of the main pillars supporting the modern theory of Mathematical Finance is the First Fundamental Theorem of Asset Pricing (henceforth FTAP) which simply states that absence of arbitrage is equivalent to the existence of a probability measure Q on the underlying probability space $(\Omega, \mathfrak{F}, P)$ which is equivalent to P and under which the price process S is a martingale.¹ In the usual setting, the stochastic process $S = (S_t)_{0 \leq t \leq T}$ adapted to the filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{0 \leq t \leq T}, P)$ is assumed to model the price evolution of d risky stocks. Moreover, it is denoted in discounted terms, i.e. one fixes a traded asset (the "bond") as numéraire and expresses stock prices S in units of this bond. Thus the central issue is whether it is possible to find such a probability measure Q , equivalent to P , under which S is a martingale.

A few words are here required to clarify the term "martingale". Before providing a formal definition of this concept let us give here some intuition. Suppose a simple mathematical model of a financial market with two dates: today and tomorrow and two assets: a riskless bond and a risky stock. For simplicity we assume interest rate equal to zero, so the price of the bond equals 1 today and tomorrow (i.e. $B_0 = B_1 = 1$). In the case of the stock we know its price today, say $S_0 = 1$, but since it is risky we do not know its value tomorrow. We model the uncertainty stochastically by defining S_1 to be a random variable depending on the state of the world tomorrow $\omega \in \Omega$. Suppose that Ω consist of only two states: good and bad, with the same probability $1/2$, and that the stock price tomorrow can be 2 in the good state or $1/2$ in the bad state. Suppose now that we want to price a new risky instrument, a call option on the stock with strike price $X = 1$. Since the price of the option tomorrow depends on the price of the stock, it can be 1 or 0 depending on the state of the world.

Now, notice that a portfolio consisting of a long position in $2/3$ shares of the stock and a short position in $1/3$ units of the bond will exactly replicate the payoff of the option tomorrow, i.e. in the good state it will pay out 1, and 0 in the bad state. The common technique of pricing by "no-arbitrage" implies that the unique option price today that is consistent with the no-arbitrage assumption has to be exactly equal to the price of the replicating portfolio, i.e. $2/3 * 1 - 1/3 * 1 = 1/3$. If not, one could make a riskless profit by buying the cheaper and selling the more expensive one.

Another classical approach to pricing contingent claims, having its origins in the actuarial methods, consists in taking expectations. By calculating the expectation with respect to the probability measure P we would obtain the option price equal to $C_0 = E(C_1) = 1/2 * 1 + 1/2 * 0 = 1/2$ which is different from the unique no-arbitrage price obtained above. What is the reason of this inconsistency? The rationale behind taking the expected value as the price of a contingent claim follows from the assumption that in the long run the buyer of a risky asset will neither gain nor lose on average. In financial terms it means that the performance of an investment in risky assets would on average equal the performance of the riskless asset, which although reasonable in actuarial setting, seems to be counter intuitive in finance, where one expects that an investment in a risky asset should, on average, yield a better performance than an investment in the bond. In our example, we have that $E(S_1) = 1.25 > 1 = E(B_1)$, so that on average the stock performs better than the bond.

Now let us look at this problem in a different way. Suppose that the world is governed by a probability Q different from P that assigns different weights to the good and bad state, and such that under this new probability the expected return of the stock equals that of the bond. It turns out that the probability measure assigning $1/3$ probability to the good state and $2/3$ to the bad state is the only solution satisfying $E_Q(S_1) = S_0 = 1$, i.e. the process S is a martingale under Q , and Q is a martingale measure for S . Notice that under this measure Q the price of the call option is exactly equal to the unique price consistent with the no arbitrage assumption.

¹In this paper we focus on the "First Fundamental Theorem of Asset Pricing" which is different from the result called the "Second Fundamental Theorem of Asset Pricing" that establishes the equivalence between the uniqueness of the martingale measure and the completeness of the pricing model.

The formal definition of the concept of martingale is provided in Definition 7 of this paper.

After these explanations it should become clear why FTAP constitutes such a crucial result in Mathematical Finance. If we are able to establish this result under some meaningful assumptions, such as no-arbitrage, no free lunch, etc., we can find martingale measure which can be used to determine prices of new assets by simply taking expectations. This is also the reason for the term "*fundamental*" in the name of this theorem.

In this paper we review the most important advances in the literature on FTAP and formulate, in precise mathematical terms, different versions of the result that has become a landmark in the research in financial theory over the last 30 years. The paper is organized as follows. Section 2 describes early works that have provided ground for the whole history of research in the development of FTAP. In this Section we discuss different versions of FTAP in the most basic case of market with no frictions, with finite time, finite set of states of the world and a finite number of assets. Section 3 reviews the studies that establish FTAP by replacing "absence of arbitrage" by conditions such as "no free lunch", "no free lunch with bounded risk" or "no free lunch with vanishing risk" that have allowed for further extensions of FTAP to more general settings, such as infinite time and infinite number of states of the world. Section 4 discusses the most important findings in the literature on FTAP in the case of markets with frictions. In Section 5 we will provide some very recent results that apply the approach of projective systems of probability measures to establish FTAP-like results in case of infinite time or infinite number of assets without drawing on free lunches. Finally, Section 6 concludes.

2. EARLY VERSIONS OF FTAP.

The origins of FTAP and other crucial results in Mathematical Finance are inseparably linked to the work of Black and Scholes (1973) and Merton (1973). What is now known as the Black-Scholes model based on the proposed by Samuelson (1965) geometric Brownian motion $S = (S_t)_{0 \leq t \leq T}$ provides a well known technique for pricing options and other derivatives. In this framework one changes the underlying measure P to an equivalent measure Q under which the discounted stock price process is a martingale. Subsequently, one prices derivatives by simply taking expectations with respect to this risk-neutral or martingale measure Q . The field of studies on FTAP has stemmed in a natural way from the need of further understanding of the concept of pricing by "no-arbitrage" and the pricing technique consisting in calculating expectations with respect to a martingale measure Q .

The very first studies that investigate these issues and establish some kind of relationship between the absence of arbitrage and the existence of pricing rules, risk neutral probabilities or martingale measures are Ross (1976, 1977, 1978) and Cox and Ross (1976) (see, for example, the survey paper Schachermayer (2009a)). Ross fixes a topological, ordered vector space (X, τ) and models the possible cash-flows at a fixed time horizon T . For example, $X = L^p(\Omega, \mathfrak{F}, P)$ where $1 \leq p \leq \infty$ and $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{0 \leq t \leq T}, P)$ is the underlying filtered probability space. If $S = (S_t)_{0 \leq t \leq T}$ is a stock price process, the set of marketed claims $M \subseteq X$ would be the set of all the outcomes of an initial investment $x \in \mathbb{R}$ plus the result of trading according to a predictable trading strategy $H = (H_t)_{0 \leq t \leq T}$. Thus a marketed claim $m \in M$ would be given by

$$m = x + \int_0^T H_t dS_t \quad (1)$$

and would be priced by setting $\pi(m) = x$.

In the case of d risky assets the process S would be \mathbb{R}^d -valued and the marketed claim would be given by

$$m = x + \int_0^T \sum_{i=1}^d H_t^i dS_t^i.$$

Much of the early literature on FTAP focused on the discussion of the precise admissibility conditions which should be imposed on the stochastic integral (1), in order to make sense both from the mathematical and economical point of view (e.g. Harrison and Kreps (1979)). Ross (1978) leaves this issue aside and simply introduces a modeling assumption that the subset $M \subseteq X$ and the pricing operator $\pi : M \rightarrow \mathbb{R}$ are given.

In this setting the concept of arbitrage can be formalized as follows:

Definition 1. We say that the *no-arbitrage condition holds* if for $m \in M$ such that $m \geq 0$, $P - a.s.$ and $P(m > 0) > 0$, we have $\pi(m) > 0$.

In other words, it is not possible to find a claim $m \in M$ which bears no risk, yields some gain with strictly positive probability and such that its price is less than or equal to zero.

An interesting question that here arises is whether it is possible to extend the pricing operator $\pi : M \rightarrow \mathbb{R}$ to a continuous, non-negative, linear functional $\pi^* : X \rightarrow \mathbb{R}$. Ross (1978) is the first study which investigates this issue and its relationship to the existence of martingale measures. Suppose $X = L^p(\Omega, \mathfrak{F}, P)$ for some $1 \leq p < \infty$, the price process $S = (S_t)_{0 \leq t \leq T}$ satisfies $S_t \in X$ for each $0 \leq t \leq T$ and M is the set of marketed claims of the form

$$m = x + \sum_{i=1}^n H_i(S_{t_i} - S_{t_{i-1}}). \quad (2)$$

where $x \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_n = T$ and $(H_i)_{i=1}^n$ is a bounded predictable (i.e. H_i is $\mathfrak{F}_{t_{i-1}}$ -measurable) process. The sums in (2) are Riemman sums corresponding to the stochastic integral (1).

Now suppose that the functional π defined for the claims of the form (2) can be extended to a continuous, non-negative functional π^* defined on $X = L^p(\Omega, \mathfrak{F}, P)$. If such an extension exists, it is induced by some function $g \in L^q(\Omega, \mathfrak{F}, P)$, where $1/p + 1/q = 1$. The non-negativity of π^* is equivalent to $g \geq 0$, $P - a.s.$ and the fact that $\pi^*(1) = 1$ implies that g is the density of a probability measure Q with Radon-Nikodym derivative $dQ/dP = g$. So, if such an extension π^* of π exists, one finds a probability measure Q on $(\Omega, \mathfrak{F}, P)$ for which

$$\pi^*\left(\sum_{i=1}^n H_i(S_{t_i} - S_{t_{i-1}})\right) = E_Q\left(\sum_{i=1}^n H_i(S_{t_i} - S_{t_{i-1}})\right)$$

for every bounded predictable process $H = (H_i)_{i=1}^n$. This is equivalent to $S = (S_t)_{0 \leq t \leq T}$ being a martingale (see the results established in Harrison and Kreps (1978) or Delbaen and Schachermayer (2006)).

Ross (1978) establishes the existence of such an extension π^* by applying the Hahn-Banach theorem which is a well known mathematical tool for separating hyperplanes. In order to be able to do so, he suggests to endow X with a topology strong enough to insure that the positive orthant $\{x \in X | x > 0\}$ is an open set. Hence two important cases arise: 1) when the probability space Ω is finite then $X = L^p(\Omega, \mathfrak{F}, P)$ is finite dimensional and its topology does not depend on $1 \leq p \leq \infty$ and 2) if $(\Omega, \mathfrak{F}, P)$ is infinite and $X = L^\infty(\Omega, \mathfrak{F}, P)$ endowed with the topology induced by the norm $\|\cdot\|_\infty$.

In this setting one can identify two convex sets to be separated: $\{m \in M : \pi(m) \leq 0\}$ and the interior of the positive cone of X . These sets are disjoint if and only if the no arbitrage condition holds. Since one can always separate an open convex set from a disjoint convex set, one can find a functional $\tilde{\pi}$ which is strictly positive on the interior of the positive cone of X and takes non-positive values on $\{m \in M : \pi(m) \leq 0\}$. The extension π^* can be obtained by normalizing $\tilde{\pi}$, i.e. the extension π^* is given by $\pi^* = \tilde{\pi}(1)^{-1} \tilde{\pi}$.

Ross' (1978) result is considered to be the first precise version of FTAP. However, it is worth to note that there are serious limitations of this result. First, in the case of infinite $(\Omega, \mathfrak{F}, P)$ the result only applies to $L^\infty(\Omega, \mathfrak{F}, P)$ endowed with the norm topology. In this case the continuous linear functional π^* is only in $L^\infty(\Omega, \mathfrak{F}, P)^*$ and not necessarily in $L^1(\Omega, \mathfrak{F}, P)$, i.e. one cannot be sure that π^* is induced by a probability measure Q , since it may happen

that $\pi^* \in L^\infty(\Omega, \mathfrak{F}, P)^*$ has also a singular part. Second limitation, which already arises in the case of finite-dimensional Ω (a case in which π^* is certainly induced by some measure Q with $dQ/dP = g \in L^1(\Omega, \mathfrak{F}, P)$) is related to the fact that one cannot be sure that the function g is strictly positive $P - a.s.$, i.e. that Q is equivalent to P .

After Ross (1978) major advances in the development of FTAP were achieved in three seminal papers: Harrison and Kreps (1979), Harrison and Pliska (1981) and Kreps (1981). Harrison and Kreps (1979) can be considered a landmark in this field. In this paper, the authors use a similar setting as Ross (1978), i.e. an ordered topological vector space (X, τ) and a linear functional $\pi : M \rightarrow \mathbb{R}$, where M is a linear subspace of X . They also deal with a finite probability space Ω . Under assumptions on the convexity and continuity of agents' preferences, they establish the equivalence between the existence of a linear, continuous and strictly positive extension $\pi^* : X \rightarrow \mathbb{R}$ and the viability of (M, π) as a model of economic equilibrium.

In subsequent paper, Harrison and Pliska (1981) provide a precise version of the theorem in terms of equivalent martingale measures in the simple setting of finite Ω and finite discrete time:

Theorem 2. *Suppose the stochastic process $S = (S_t)_{t=0}^T$ is based on a finite, filtered, probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t=0}^T, P)$. The market model contains no arbitrage possibilities if and only if there is an equivalent martingale measure for S .*

Historically, considerations similar to Harrison and Pliska (1981) appeared in works on symbolic logic of Shimony (1955) and Kemeny (1955) who showed that in the case with finitely many states of the world, a family of possible bets does not allow for making a riskless profit, if and only if there exists a probability measure Q which prices the possible bets by taking conditional Q -expectations.

Despite the beauty of the Harrison and Pliska (1981) result one has to realize the severity of the restriction of finite Ω . For example, continuous time models would obviously require working with infinite probability spaces. Further extension of FTAP was obtained in Dalang et al. (1990) where the authors show that for an \mathbb{R}^d -valued process $(S_t)_{t=0}^T$ in finite discrete time the no arbitrage condition is equivalent to the existence of an equivalent martingale measure. The Dalang-Morton-Willinger theorem is widely recognized as one of the most important results in the arbitrage pricing theory as it is a deep generalization of the Harrison-Pliska theorem which has exactly the same formulation but under assumption of finite probability space. Different proofs of the Dalang-Morton-Willinger theorem were provided in Schachermayer (1992), Rogers (1994), Kabanov and Kramkov (1994), Jacod and Shiryaev (1998), Kabanov and Stricker (2001). Subsequent intensive mathematical studies on FTAP generalization (see, for example the survey paper Kabanov (2001)) reveal that in more general settings: with infinite time, infinite number of assets or in the presence of market frictions a simple version of FTAP cannot be proved, i.e. the absence of arbitrage is not sufficient to construct martingale measures or risk neutral probability measures under which the price process is a martingale. Interesting counter-examples showing the absence of equivalent martingale measures in the arbitrage-free markets in the case of markets with infinite number of assets or with infinite set of trading dates were provided by Back and Pliska (1991) and Schachermayer (1992).

3. FTAP AND THE GENERALIZED ARBITRAGE.

The difficulty of proving FTAP beyond the simple framework forced researchers to search for other means that would allow to overcome this problem and establish the existence of martingale measures. Further extensions of FTAP are obtained by generalizing the concept of arbitrage and establishing the existence of equivalent martingale measures by the condition of "no free lunch", "no free lunch with bounded risk" or "no free lunch with vanishing risk". These new advances in the development of FTAP were possible thanks to a breakthrough result of Kreps (1981).

In the setting described in the previous section, Kreps (1981) considers the set of marketed claims $M \subseteq X$ and a linear functional $\pi : M \rightarrow \mathbb{R}$. X is now $X = L^p(\Omega, \mathfrak{F}, P)$, $1 \leq p \leq \infty$

equipped with the topology τ of convergence in norm or if $X = L^\infty(\Omega, \mathfrak{F}, P)$ equipped with the Mackey topology induced by $L^1(\Omega, \mathfrak{F}, P)$. This setting ensures that a continuous linear functional on (X, τ) will be induced by a measure Q which is absolutely continuous with respect to P . Let $M_0 = \{m \in M : \pi(m) = 0\}$. In this setting the no arbitrage condition holds if

$$M_0 \cap X_+ = \{0\},$$

where X_+ is the positive orthant of X . In order to obtain an extension of π to a continuous, linear functional $\pi^* : X \rightarrow \mathbb{R}$ one has to find an element in $(X, \tau)^*$ which separates the convex set M_0 from the disjoint convex set $X_+ \setminus \{0\}$. Kreps (1981), uses a version of the Hahn-Banach theorem to separate these sets. For that he defines

$$A = \overline{M_0 - X_+}, \quad (3)$$

where the bar denotes the closure with respect to the topology τ and A is required to satisfy the following condition called by Kreps (1981) "no free lunch property" (NFL):

$$A \cap X_+ = \{0\}. \quad (4)$$

Definition 3. *The financial market defined by (X, τ) , M and π admits a free lunch if there are nets $(m_\alpha)_{\alpha \in I} \in M_0$ and $(h_\alpha)_{\alpha \in I} \in X_+$ such that*

$$\lim_{\alpha \in I} (m_\alpha - h_\alpha) = x \quad (5)$$

for some $x \in X_+ \setminus \{0\}$.

It is worth to note that free lunch is a more general concept than arbitrage, thus "NFL" condition by being a stronger assumption than "no-arbitrage" allows Kreps (1981) to establish further extension of FTAP in a more general setting:

Theorem 4. *Let $(\Omega, \mathfrak{F}, P)$ be countably generated and $X = L^p(\Omega, \mathfrak{F}, P)$ endowed with the norm topology τ if $1 \leq p < \infty$, or the Mackey topology induced by $L^1(\Omega, \mathfrak{F}, P)$ if $p = \infty$. Let $S = (S_t)_{0 \leq t \leq T}$ be a stochastic process taking values in X . Define $M_0 \subseteq X$ to consist of the simple stochastic integrals $\sum_{i=1}^n H_i(S_{t_i} - S_{t_{i-1}})$ as in (2). Then the NFL condition (3) is satisfied if and only if there is a probability measure Q with $dQ/dP \in L^q(\Omega, \mathfrak{F}, P)$, where $1/p + 1/q = 1$ such that $S = (S_t)_{0 \leq t \leq T}$ is a Q -martingale.*

Kreps' (1981) results set new standards for further research on the extensions of FTAP in more general frameworks. A detailed description of a rather extensive literature in this field can be found in Delbaen and Schachermayer (2006). For the case $1 \leq p \leq \infty$ of Kreps' theorem it is worth to mention Duffie and Huang (1986) or Stricker (1990). This last one using the theorem of Yan (1980) provides a different proof of Kreps' theorem which does not require the assumption that $(\Omega, \mathfrak{F}, P)$ is countably generated.

Further studies discuss whether one can replace the use of nets in Kreps (1981) by infinite sequences. Dealbean (1992) showed in the case of continuous processes $(S_t)_{0 \leq t \leq T}$ that the answer is "yes" if one replaces the deterministic times $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ in (2) by stopping times $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n = T$. Schachermayer (1994) provides a positive answer to this question in the case of processes $(S_t)_{t=0}^\infty$ in infinite, discrete time. In the above two cases one can equivalently replace the NFL condition of Kreps by the "no free lunch with bounded risk" (NFLBR), in which, in addition to (5) one imposes that $(\|m_\alpha - h_\alpha\|_\infty)_{\alpha \in I}$ is bounded, so there is a constant $M > 0$ such that $m_\alpha \geq -M, P$ -a.s. for each $\alpha \in I$ which explains "bounded risk". In spite of the result of Schachermayer (1994), Delbaen and Schachermayer (1994) provide a counter-example which shows that in the case of general semi-martingale models NFLBR does not necessarily imply the existence of an equivalent martingale measure. Hence, at this point

it becomes clear that it is impossible to obtain further extensions of Krep’s theorem to more general settings by only using simple integrals.

As Schachermayer (2009) argues: “in order to obtain sharper results, one has to go beyond the framework of simple integrals and rather use general stochastic integrals (1). After all, the simple integrals only are a technical gimmick, analogous to step functions in measure theory. In virtually all the applications, e.g., the replication strategy of an option in the Black-Scholes model, one uses general integrals...p. 10”

The integration theory suggests that for the general integrands to be mathematically well-defined, the price process $S = (S_t)_{0 \leq t \leq T}$ should be a semi-martingale. This seems to be also justified from the economic point of view: Delbaen and Schachermayer (1994) show that for a locally bounded stochastic process, a very weak form of Krep’s NFL condition involving simple integrands (2), already implies that S is a semi-martingale.

Following these arguments, further extensions of FTAP are based on this assumption. Delbaen and Schachermayer (1994) establish a version of FTAP for the price process being a semi-martingale by introducing the concept of “free lunch with vanishing risk”. For that, they first impose additional admissibility condition, i.e. for S-integrable predictable process $H = (H_t)_{0 \leq t \leq T}$ there exists a constant $M > 0$ such that $\int_0^t H_u dS_u \geq -M$, a.s., for $0 \leq t \leq T$. Next, they define a set of random variables uniformly bounded from below:

$$K = \left\{ \int_0^T H_t dS_t \mid H \text{ is admissible} \right\}$$

and a set of bounded random variables:

$$C = \{g \in L^\infty(\Omega, \mathfrak{F}, P) : g \leq f \text{ for some } f \in K\}$$

$$= [K - L_+^0(\Omega, \mathfrak{F}, P)] \cap L^\infty(\Omega, \mathfrak{F}, P),$$

where $L_+^0(\Omega, \mathfrak{F}, P)$ is the set of non-negative measurable functions. Finally, they introduce the concept of “free lunch with vanishing risk” as follows:

Definition 5. A locally bounded semi-martingale $S = (S_t)_{0 \leq t \leq T}$ satisfies the **no free lunch with vanishing risk (NFLVR)** condition if

$$\bar{C} \cap L_+^\infty(\Omega, \mathfrak{F}, P) = \{0\}$$

where \bar{C} denotes the $\|\cdot\|_\infty$ -closure of C .

The condition of NFLVR is weaker than the Krep’s NFL and stronger than the no-arbitrage condition as in the initial works on FTAP. Using the NFLVR condition Delbaen and Schachermayer (1994) establish the following version of FTAP:

Theorem 6. Let $S = (S_t)_{0 \leq t \leq T}$ be a locally bounded real-valued semi-martingale. There exists a probability measure Q on (Ω, \mathfrak{F}) equivalent to P under which S is a local martingale if and only if S satisfies the condition of no free lunch with vanishing risk.

It is worth to note that Theorem 6 still has an important limitation of generality as it assumes that the price process is locally bounded. If one deals with continuous price processes this condition is satisfied, however, this assumption would not hold for processes with jumps.

The case of general semi-martingales without any boundedness assumption was analyzed, for example, in Delbaen and Schachermayer (1994,1998). However, the concept of local martingales has to be further weakened to sigma-martingales. A detailed discussion on differences between martingales, local martingales and sigma-martingales can be found in the survey paper Schachermayer (2009b). The following definition summarizes all three concepts:

Definition 7. An \mathbb{R}^d -valued stochastic process $S = (S_t)_{0 \leq t \leq T}$ based on and adapted to $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{0 \leq t \leq T}, P)$ is called a

(i) **martingale** if

$$E[S_t | \mathfrak{F}_u] = S_u,$$

$$0 \leq u \leq t \leq T.$$

(ii) **local martingale** if there exists a sequence $(\tau_n)_{n=1}^\infty$ of $[0, T] \cup \{+\infty\}$ -valued stopping times, increasing a.s. to ∞ , such that the stopped processes $S_t^{\tau_n}$ are all martingales, where

$$S_t^{\tau_n} = S_{t \wedge \tau_n},$$

$$0 \leq t \leq T.$$

(iii) **sigma-martingale** if there is an \mathbb{R}^d -valued martingale $M = (M_t)_{0 \leq t \leq T}$ and a predictable M -integrable \mathbb{R}_+ -valued process φ such that $S = \varphi \cdot M$.

The process $\varphi \cdot M$ is defined as the stochastic integral in the sense of semi-martingales:

$$(\varphi \cdot M)_t = \int_0^t \varphi_u dM_u,$$

$$0 \leq t \leq T.$$

It can be shown that a local martingale is a sigma-martingale, so that (i) \Rightarrow (ii) \Rightarrow (iii) is true but the reverse implications fail to hold.

Delbaen and Schachermayer (1994) establish the following version of FTAP for the sigma-martingales:

Theorem 8. *Let $S = (S_t)_{0 \leq t \leq T}$ be an \mathbb{R}^d -valued semi-martingale. There is a probability measure Q on (Ω, \mathfrak{F}) equivalent to P under which S is a sigma-martingale if and only if S satisfies the condition of NFLVR with respect to admissible strategies.*

Further work on FTAP involves considerations whether it is possible to formulate a version of FTAP which does not rely on the concepts of local or sigma-martingales but rather on "true" martingales. For instance, Yan (1998) applies a change of numéraire, weakens Delbaen and Schachermayer (1994) admissibility condition by replacing it by allowance condition (i.e. a predictable S -integrable process is allowable if $\int_0^t H_u dS_u \geq -M(1 + S_t)$ a.s., for $0 \leq t \leq T$) and obtains a version of FTAP under the NFLVR assumption for the stochastic process S being a positive semi-martingale.

4. FTAP IN THE PRESENCE OF MARKET FRICTIONS.

Most of the models in Mathematical Finance assume absence of frictions in the markets. Transaction costs and the presence of bid and ask spreads are probably the most prevailing frictions in the real markets. Since bid-ask spreads are usually a small fraction of prices the theory of frictionless markets relies on the assumption that small frictions have small effects. However, it turns out that many results in the no-arbitrage and superreplication theory have completely different solutions in the presence of transaction costs and, in fact, do not converge to the frictionless solution as the bid-ask spread shrinks to a single price. This kind of problems have been analyzed, for example, by Cherny (2007), Cvitanic et al. (1999), Jakubenas et al. (2003), Levental and Skorohod (1997) or Soner et al. (1995). These findings gave a ground for extensive studies on the models with frictions and, in particular, on the extensions of FTAP to the imperfect markets case.

A pioneering work in this field was the paper of Jouini and Kallal (1995) who characterize the absence of arbitrage (in the form of free lunch - a generalized arbitrage) in the presence of bid-ask spreads. Bid-ask spreads usually reflect the existence of transaction costs but can be also generated by a variety of other frictions and investment constraints. Their main result states that the market is arbitrage-free if and only if there exists at least an equivalent probability measure that transforms some process lying between the bid and ask price processes into a martingale (after a normalization). The set of martingale measures provides pricing rules that allow to price new assets, i.e. the price of a contingent claim will be equal to its largest expected value with respect to the martingale measures. Moreover, the largest and the smallest expected values of

the contingent claim with respect to the martingale measure define the tightest bounds on its price consistent with the no-arbitrage assumption. In order to provide the formal version of their result let us first introduce their setting.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $X = L^2(\Omega, \mathfrak{F}, P)$ the space of square integrable random variables on $(\Omega, \mathfrak{F}, P)$ assumed to be separable. Let Ψ be the set of linear functionals on X . The set of marketed claims is denoted by M and is assumed to be a convex cone of X . π is a sublinear price functional defined on M . A contingent claim $m \in M$ can be bought at price $\pi(m)$ and sold at $-\pi(-m)$. So defined price functional reflects the existence of transaction costs, or bid ask spreads, since $\pi(m) \geq -\pi(-m)$, i.e. the agent pays more to buy the claim m than what he can receive from selling it.

Jouini and Kallal (1995) result states:

Theorem 9. (i) *The securities price model admits no multiperiod free lunch if and only if there exists at least a probability measure P^* equivalent to P with $E((dP^*/dP)^2) < \infty$ and a process Z^* satisfying $Z' \leq Z^* \leq Z$ such that Z^* is a martingale with respect to the filtration $\{\mathfrak{F}_t\}$ and the probability measure P^* .*

(ii) *Moreover, if we denote by E^* the expectation operator associated to P^* , there is a one-to-one correspondence between the set of such expectation operators and the set of linear functionals $\psi \in \Psi$ such that $\psi|_M \leq \pi$. This correspondence is given by the formulas*

$$P^*(B) = \psi(\mathbf{1}_B),$$

for all $B \in \mathfrak{F}$ and $\psi(x) = E^*(x)$, for all $x \in X$.

(iii) *Furthermore, for all $m \in M$ we have*

$$[-\pi(-m), \pi(m)] = Cl\{E^*(m) : P^* \text{ is a martingale measure}\}.$$

Subsequent works, for example, Cvitanić and Karatzas (1996), Kabanov (1999), Kabanov et al. (2003), Kabanov and Stricker (2002), Schachermayer (2004), Guasoni et. al (2010) develop Jouini and Kallal (1995) results and establish the equivalence between absence of arbitrage and the existence of consistent price systems (CPSs) in different settings. CPSs in markets with transaction costs play the same role as martingale measures in frictionless markets. A CPS is simply a shadow frictionless asset, which admits a martingale measure such that its price is always within the bid-ask spread of the original asset. The existence of CPSs links the problems of no-arbitrage and superreplication to their frictionless counterparts, where solutions are well-understood.

5. FURTHER FTAP EXTENSIONS - PROJECTIVE SYSTEMS APPROACH.

In the previous sections we have seen that in general settings a simple version of FTAP cannot be proved since the absence of arbitrage is not sufficient to construct martingale measures. Accordingly, the literature constructs martingale measures by generalizing the concept of arbitrage to free lunch, free lunch with bounded risk or free-lunch with vanishing risk. Free-lunch can be understood as an “approximated arbitrage” in the sense that it is “quite close” to an arbitrage portfolio. However, arbitrage seems to be a more intuitive concept, with clearer economic interpretation and easier to be tested empirically. It is also worth to recall that the classical asset pricing models (binomial model, Black and Scholes model, etc.) usually deal with the concept of arbitrage and not free lunch. Following these arguments, some recent studies try to avoid the use of free lunch and retrieve the concept of arbitrage to establish FTAP-like results in more general settings. For example, in the markets with infinite assets, the existence of martingale measures can also be established by the theory of large financial markets (e.g. Kabanov and Kramkov (1998) and Klein (2000)). Each “small” market is arbitrage-free and there is an equivalent martingale measure on each of the small market. Still there can be various

forms of approximate arbitrage opportunities when one considers the sequence of markets, and the notion of "no arbitrage" is generalized to be sufficient to get a risk neutral measure for the large financial market.

Recently, Balbás *et al.* (2002), Balbás and Downarowicz (2007) and Balbás and Guerra (2009) rely on the assumption of absence of arbitrage (and not absence of free lunch) to extend FTAP and provide martingale measures in the case of infinite number of trading dates and infinite number of securities. They show that it is possible to solve the counter-examples of Back and Pliska (1991) and Schachermayer (1992) without drawing on free lunches.

In order to characterize the absence of arbitrage in the case of infinite and countable set of trading dates, Balbás *et al.* (2002) build an appropriate projective system of perfect probability measures (see Musial (1980)) that are risk-neutral for each finite subset of trading dates. Then they show that the projective limit is risk-neutral for the whole set of trading dates, in the sense that the set of states of the world and the price process may be extended to a "new price process" which is a martingale under this projective limit. The initial probability measure and the risk-neutral one cannot be equivalent, as illustrated in the counter-example of Back and Pliska (1991). However, for any finite subset of trading dates one can find projections of both measures that are equivalent, and there are Radon-Nikodym derivatives in both directions. Balbás *et al.* (2002) use this property to introduce the concept of "projective equivalence" of probability measures. Balbás and Guerra (2009) addresses the equivalence between the absence of arbitrage and the existence of equivalent martingale measures in a more general setting where the set of trading dates may be finite or countable, with bounded or unbounded horizon.

In the same line, Balbás and Downarowicz (2007) draw on the projective system approach in order to establish the equivalence between the absence of arbitrage and the existence of projectively equivalent martingale measures, which provide pricing rules allowing for the valuation of new assets. In contrast to Balbás *et al.* (2002) the results are established by means of projective limits of projective systems of Radon probability measures (see Schwartz (1973)) rather than perfect measures. Their analysis is quite general since they do not impose any conditions on the set of assets or on the properties of future prices. Finally, they show that the equivalence holds for many significant cases like complete or finitely generated markets. Projectively equivalent pricing rules can also be found for more complex markets. Under some regularity properties, only the possibility of pricing new securities is necessary and sufficient. The projective system approach allows to enlarge the set of states of nature and to identify this set with the set of real prices. Thus a complete equivalence between the initial probability measure and the martingale measure does not hold in general. However, the existence of densities between real probabilities and risk-neutral ones is guaranteed by introducing the concept of projective equivalence, which implies that both the martingale measure and the initial probability measure generate equivalent projections.

6. CONCLUSIONS

In this survey article we have reviewed the extensive literature on the Fundamental Theorem of Asset Pricing (FTAP) which can be considered one of the main pillars of the modern theory of Mathematical Finance. It establishes the equivalence between the absence of arbitrage and the existence of an equivalent probability measure under which the price process is a martingale. It turns out, however, that only in the simplest setting of markets without frictions, finite time, finite number of states of the nature and finite number of assets such result holds without the need of introducing sophisticated assumptions. In more general settings, a simple version of FTAP cannot be proved since the absence of arbitrage is not sufficient to construct martingale measures. Accordingly, the literature overcomes this failure and constructs martingale measures by generalizing the concept of arbitrage to free lunch, free lunch with bounded risk or free lunch with vanishing risk. Following this approach FTAP for markets with frictions is established through the martingale property of a price process lying within the bid and the ask price processes. Recently, some authors try to avoid drawing on free lunches and establish FTAP-

like results in more general settings relying on the "true" no-arbitrage condition. It turns out that, by drawing on the projective systems approach and the concept of projective equivalence of probability measures, FTAP-like results can be established in quite general settings, such as infinite time of infinite number of assets.

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