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# Hedging bond portfolios against infinitely many ranked factors of risk<sup>1</sup>

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**Abstract:** The paper considers bond portfolios affected by both interest rate and default-risk. In order to guarantee a correct performance of our analysis, we will hedge against an infinite number of factors. Hence, we do not have to impose and do not depend on any assumption concerning the dynamic behavior of the term structure of interest rates. On the other hand, since a complete hedging is not feasible unless some ideal situations hold, we rank the factors according to the empirical evidence. Thus, we make the most important risks vanish, and minimize the effect of those kinds of risk less usual in practice.

**JEL Classification:** G11

**Key words:** Interest rate risk, default risk, infinite factors, hedging criteria.

**Título:** Cobertura de carteras de bonos contra infinitos factores de riesgo categorizados

**Resumen:** Este artículo considera una cartera de bonos afectada por el riesgo de tipo de interés y el riesgo de crédito. Para garantizar una correcta realización de nuestro análisis cubriremos contra un número infinito de factores de riesgo. Por tanto, no imponemos ni dependemos de ninguna hipótesis relativa al comportamiento dinámico de la estructura temporal de los tipos de interés. Por otra parte, como no es factible realizar una cobertura completa a menos que se den determinadas situaciones ideales, categorizamos los factores basándonos en la evidencia empírica. De ese modo, hacemos que los riesgos más importantes desaparezcan y minimizamos el efecto de aquellos tipos de riesgos menos usuales en la práctica.

**JEL Classification:** G11

**Palabras Clave:** Riesgo de tipo de interés, riesgo de crédito, infinitos factores, criterios de cobertura.

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## 1. INTRODUCTION

Insurance companies often hedge a significant part of their liabilities by purchasing credit-risk free fixed income securities. Both the legal constraints and the low volatility of these assets make it convenient to draw on them when the risk level needs to be controlled.

However, some bonds issued by private companies are becoming more and more usual in hedged portfolios. These assets do not reflect the volatility levels of shares or derivatives and they pay a risk premium that may be interesting to risk-averse traders. Whence it may be worthwhile to address hedging problems when both interest rate risk and credit risk are simultaneously combined.

If one analyzes the literature concerning credit risk free portfolios, there are no unified criteria in order to hedge the interest rate risk. So, some authors try to guarantee the highest possible return at a given date (see for instance Bierwag and Khang (1979), or Barber (1999), and references therein) while others try to minimize the sensitivity of the portfolio price with respect to any shock on the Term Structure of Interest Rates (henceforth *TSIR*). This paper will focus on the second approach.

Besides, authors do not necessarily agree when to minimize the portfolio price sensitivity either. In fact, they provide different hedging criteria owing to distinct answers to two key questions: How many factors do we need to explain the *TSIR* behavior? What are these factors? There are many important contributions on immunization whose differences are provoked by these topics. For instance, one can consider Chambers *et al.* (1988), Reitano (1992), Bierwag *et al.* (1993), Paroush and Prisman (1997), Balbás *et al.* (2002)<sup>b</sup>, etc.

Bowden (1997) and Barber and Copper (1998) are two seminal papers that prevent the risk generated by the factors choice. They consider infinite factors in such a way that every (square-integrable) function is a feasible shift on the *TSIR*. Since they yield explicit solutions to their problems, they are providing hedging in a general framework that does not depend on any dynamic (or static) assumption on the *TSIR* behavior. Unfortunately, they prove that there are no portfolios with null sensitivity with respect to infinite factors, so their optimal sensitivity is positive. Furthermore, the optimal sensitivity is attained if the real shock coincides with “the most negative scenario” on the *TSIR* (Bowden calls this scenario as Direction  $X$ ).

Elton *et al.* (1990) and Litterman and Scheinkman (1991) are very important papers pointing out the existence of finite number of factors reflecting the *TSIR* behavior. The first paper focuses on the tie between factors and spot rates, while the second one considers usual factors as level, slope, curvature, etc.

The present paper combines the idea of Bowden, Barber and Copper along with the existence of a finite number of significant factors. Consequently, we will consider an infinite number of ranked factors. We will seek for protection against the significant factors and later we will minimize the residual risk according to the Barber and Copper criterion. Hence we are retrieving a consensus between the classical and the Barber and Copper approaches. Indeed, with regard to the classical point of view, we immunize against the significant factors, although our analysis seems to provide two important advantages: First, we do not fix the number of factors before analyzing how many of them may be hedged, so we protect against as many factors as possible. Second, we also minimize the non-linked to the factors residual risk. With regard to the Barber and Copper ap-

proach, our analysis seems to reveal another advantage, since we seek immunization against the significant factors and, therefore, “our Direction  $X$ ” becomes much more unrealistic, and we are fully immunized against probable shocks. In this sense, our strategy should performance far better.

We will consider a wide setting far more general than that consisting of bonds and the *TSIR*. So we will be dealing with general securities whose prices depend on a state variable belonging to an abstract Hilbert space. This provides an additional advantage, since this broad framework also applies for non-credit-risk free securities.

The paper may be outlined as follows. Second section will introduce the basic concepts and notations and study the space of immunized portfolios. The major findings are Lemma 5 and Theorem 6, where it is proved that there are no immunized portfolios in our general context<sup>4</sup> and they are also provided those methods leading to strategies immunized against as many factors as possible.

Section 3 provides four hedging criteria. Since total immunization is frequently infeasible, Criteria 2, 3 and 4 try to minimize the residual risk. Criterion 2 draws on the Barber and Copper (1998) ideas, Criterion 3 uses more classical approaches, and Criterion 4 combines both kinds of arguments. Theorem 8 seems to be another significant finding of this section because, under convexity assumptions, it provides upper bounds for possible capital losses of hedged strategies. It is in the line of those formulas provided by the significant contributions of Fong and Vasicek (1984) and Shiu (1987, 1990), and it may be worth to remind that it also applies for infinite factors and another types of risk (credit risk, for instance).

Section 4 applies the previously developed theory. Firstly, we consider the classical immunization problem, and later we add credit risk. As said above, it is important to remark the interest of including non-credit-risk-free bonds due to their low volatility (in comparison with stocks and derivatives) and risk premium. Furthermore, their low volatility enables us to analyze them by using the introduced methodology, as an alternative to those procedures linked to the concept of risk measure.

Section 5 summarizes the article.

## 2. THE SPACE OF IMMUNIZED PORTFOLIOS

Consider  $n$  assets denoted by  $B_1, B_2, \dots, B_n$ . Suppose that the separable Hilbert space  $H$  contains the state variable  $h \in H$  generating the value of  $B_i$ ,  $i = 1, 2, \dots, n$ .<sup>5</sup> To be precise, let  $V_i : H \rightarrow \mathfrak{R}$ ,  $i = 0, 1, \dots, n$ , be the function yielding the price of  $B_i$  after the shock  $h \in H$  on the state variable, *i.e.*, we are assuming the possible existence of an exogenous change  $h \in H$ , in such a way that the initial  $h_0 \in H$  becomes  $h_0 + h$ , and the security prices vary from  $V_i(0)$  to  $V_i(h)$ ,  $i = 1, 2, \dots, n$ . Obviously  $p_i = V_i(0)$  is the initial (previous to the shock) price of  $B_i$ ,  $i = 1, 2, \dots, n$ . We will suppose that  $V_i$  is Fréchet differentiable,  $i = 1, 2, \dots, n$ .

In further sections we will provide several examples adapted to the general framework above, though we can present now the most usual one. So, suppose that  $B_1, B_2, \dots, B_n$  are bonds.<sup>6</sup> Assume

4 Unless ideal assumptions hold.

5 Those properties related to Hilbert spaces may be found, for instance, in Maurin (1972).

6 The theory still applies if some  $B_i$  are interest rate-linked forward or future contracts.

that  $T$  is a future date such that all of the bond maturities lie within the time interval  $[0, T]$ . The initial  $TSIR$  will be an element  $h_0 \in L^2[0, T]$ , Hilbert space of the real-valued square-integrable functions on  $[0, T]$ . In addition, we can assume that any feasible shock  $h$  on the  $TSIR$  also belongs to  $H = L^2[0, T]$ . Furthermore, for  $i = 1, 2, \dots, n$ ,  $V_i(h)$  is easily obtained by discounting the coupons of  $B_i$  with the  $TSIR$   $h_0 + h$  (see Section 4 for further details). This particular case will be referred to as the “Classic Immunization Problem” (henceforth,  $CIP$ ).

The vector  $q = (q_1, q_2, \dots, q_n) \in \mathfrak{R}^n$  will represent the portfolio composed of  $q_i$  units of  $B_i$ ,  $i = 1, 2, \dots, n$ . Clearly,  $V : \mathfrak{R}^n \times H \rightarrow \mathfrak{R}$ , given by

$$V(q, h) = \sum_{i=1}^n q_i V_i(h),$$

is the price of  $q$  after the shock  $h$ , and

$$V(q, 0) = \sum_{i=1}^n q_i V_i(0) = \sum_{i=1}^n q_i p_i$$

is the initial price of  $q$ .  $V$  is obviously linear in the  $q$  – variable, and differentiable with respect to  $h$ . These properties will play an important role in order to reach our major objective, *i.e.*, the “minimization” of the sensitivity

$$|V(q, h) - V(q, 0)|,$$

which will be approximated by the first differential of

$$V_q = V(q, -) : H \rightarrow \mathfrak{R}$$

at  $0 \in H$ .

According to the Riesz Representation Theorem, the differential of  $V_q = V(q, -)$  at  $0 \in H$  may be identified with a unique vector

$$\partial V_q \in H,$$

usually called the gradient of  $V_q$  at  $0$ . Consequently, the derivative of  $V_q$ , with respect to  $u \in H$  at  $0 \in H$ , will be given by

$$D_u V_q = \langle \partial V_q, u \rangle \in \mathfrak{R}, \tag{0.1}$$

scalar (or inner) product of  $\partial V_q$  and  $u$ .

**Definition 1** A portfolio  $q$  is said to be immunized with respect to  $u \in H$  if  $D_u V_q = 0$ . Strategy  $q$  is said to be immunized if it is immunized with respect to every  $u \in H$ .

Recall that  $D_u V_q$  is called directional derivative, as long as

$$\|u\|^2 = 1.$$

According to (0.1),  $q$  is immunized with respect to those vectors orthogonal to  $\partial V_q$ . Remember that these vectors compose a closed hyperplane of  $H$ , unless  $\partial V_q$  vanishes. If so,  $q$  is immunized. Otherwise, the Cauchy-Schwartz inequality and (0.1) trivially show that the highest directional derivative of  $V_q$  is achieved if  $u$  and  $\partial V_q$  are proportional, in which case it attains the value  $\|\partial V_q\|$ . Summarizing, one has the following result:

**Proposition 1** *The following equality holds for every  $q \in \mathfrak{R}^n$ .*

$$\text{Max}_{\{u \in H, \|u\| \leq 1\}} |D_u V_q| = \|\partial V_q\|$$

Consequently,  $q$  is immunized if and only if  $\|\partial V_q\| = 0$  (or, equivalently,  $\partial V_q = 0 \in H$ ). Moreover, if the maximum above does not vanish, then it is achieved at  $u = \frac{\partial V_q}{\|\partial V_q\|}$ .

Let  $\{f_r\}_{r=0}^\infty$  be an orthonormal basis of  $H$ . It is known that

$$h = \sum_{r=0}^\infty h_r f_r \tag{1.2}$$

is the Fourier representation of  $h \in H$ , with  $h_r \in \mathfrak{R}$  being the coefficient

$$h_r = \langle h, f_r \rangle, \tag{1.3}$$

for every  $r \in \mathbb{N}$  and every  $h \in H$ . If  $h \in H$  has the representation above and  $h^* = \sum_{r=0}^\infty h_r^* f_r \in H$ , then

$$\langle h, h^* \rangle = \sum_{r=0}^\infty h_r h_r^*. \tag{1.4}$$

In particular, Parsevals equality establishes that

$$\|h\|^2 = \sum_{r=0}^\infty h_r^2 \tag{1.5}$$

for every  $h \in H$ .

Hereafter, the orthonormal basis above may be understood as the set of “factors” or “factors of risk”. We will see several examples in further sections. At the moment we can illustrate a simple case by considering the *CIP*. So,  $H = L^2[0, T]$ , and we can build  $\{f_r\}_{r=0}^\infty$  by applying the Gram-Schmidt process to the (non-orthonormal) basis of  $H$  composed of polynomials  $\{1, t, t^2, \dots, t^r, \dots\}_{r=0}^\infty$ . In such a case,  $D_{f_0} V_q$  may be interpreted as the risk level of  $q$  against additive shocks on the *TSIR* (or the risk of  $q$  against the shock level),  $D_{f_1} V_q$  will be the risk related to the shock slope,  $D_{f_2} V_q$  will represent the risk against curvature, etc.

Next, we will present several properties of the set of immunized portfolios. All of them also hold if we deal with the *CIP* and with a finite number of factors of risk.<sup>7</sup>

**Proposition 2**  *$q \in \mathfrak{R}^n$  is immunized if and only if it is immunized against all the factors  $\{f_r\}_{r=0}^\infty$ .*

<sup>7</sup> i.e., if we deal with the *CIP* and  $L^2[0, T]$  is replaced by a finite-dimensional space, for instance, the space of polynomials of degree lower or equal to a fixed  $m$  ( $m < n$ ), as in Chambers *et al.* (1988).

**Proof.** Suppose that  $q$  is immunized against all the factors. Then, according to Definition 1 and (0.1), one has that  $(\partial V_q)_r = \langle \partial V_q, f_r \rangle = 0$  for every  $r \in \mathbb{N}$  and, therefore, Parsevals equality (1.5) leads to  $\|\partial V_q\| = 0$ . Now Proposition 1 applies.

Denote by  $\partial V_i$  the gradient of  $V_i$ ,  $i=1,2,\dots,n$ , at  $0 \in H$ . Since  $V(-,h) : \mathfrak{R}^n \rightarrow \mathfrak{R}$  is linear for every  $h \in H$ , it is easy to prove that

$$\partial V_q = \sum_{i=1}^n q_i \partial V_i \tag{2.6}$$

for every portfolio  $q = (q_1, q_2, \dots, q_n) \in \mathfrak{R}^n$ . Expression (2.6), along with Proposition 1, trivially lead to:

**Proposition 3** *The set  $I_0$  of immunized portfolios is a vector subspace of  $\mathfrak{R}^n$ .*

**Proposition 4** *Let  $q = (q_1, q_2, \dots, q_n) \in \mathfrak{R}^n$ , and  $u \in H$ . Then,  $q$  is immunized with respect to  $u$  if and only if*

$$\sum_{i=1}^n q_i \langle \partial V_i, u \rangle = 0.$$

*Therefore,  $q$  is immunized if and only if it solves the linear and homogeneous (infinite) system of equations*

$$\sum_{i=1}^n q_i \langle \partial V_i, f_r \rangle = 0 \tag{4.7}$$

$$r = 0, 1, 2, \dots$$

**Proof.** Strategy  $q$  is immunized with respect to  $u$  if and only if

$$\langle \partial V_q, u \rangle = 0.$$

Thus, (2.6) trivially leads to  $\sum_{i=1}^n q_i \langle \partial V_i, u \rangle = 0$ . The remainder of this lemma follows from Proposition 2.

Propositions 2 and 4 and System (4.7) may be significantly relaxed.

**Lemma 5** *There exists  $S \in \mathbb{N}$ , such that the following assertions hold:*

- a) *Given  $q \in \mathfrak{R}^n$ , it is immunized if and only if it is immunized with respect to  $\{f_r\}_{r=0}^S$ .*
- b) *Given  $q \in \mathfrak{R}^n$ , it is immunized if and only if it solves the linear and homogeneous (finite) system of equations*

$$\sum_{i=1}^n q_i \langle \partial V_i, f_r \rangle = 0 \tag{5.8}$$

$$r = 0, 1, 2, \dots, S.$$

**Proof.** For every  $r \in \mathbb{N}$ , consider the vector of  $\mathfrak{R}^n$  whose coordinates are the  $r^{\text{th}}$  – Fourier coefficients of  $(\partial V_i)_{i=1,2,\dots,n}$ . They compose the subset

$$L = \{(\langle \partial V_i, f_r \rangle)_{i=1,2,\dots,n} \in \mathfrak{R}^n; r = 0, 1, 2, \dots\}.$$

$L$  has an infinite number of elements, and therefore there exists  $S \in \mathbb{N}$ , such that  $(\langle \partial V_i, f_r \rangle)_{i=1,2,\dots,n}$  linearly depends on  $\{(\langle \partial V_i, f_s \rangle)_{i=1,2,\dots,n} \in \mathfrak{R}^n; s = 0, 1, 2, \dots, S\}$  whenever  $r > S$ . Whence (4.7) and (5.8) have the same set of solutions. Now, the results follow from Proposition 4.

The latter theorem does not guarantee the set

$$\{(\langle \partial V_i, f_r \rangle)_{i=1,2,\dots,n} \in \mathfrak{R}^n; r = 0, 1, 2, \dots, S\}$$

to be linearly independent. If it is dependent, then some equations of (5.8) and some elements of  $\{f_r\}_{r=0}^S$  may be eliminated in the statements above.

Suppose that  $\{u_0, u_1, \dots, u_r, \dots\}$  is a Schauder basis of  $H$  generating the orthonormal basis  $\{f_0, f_1, \dots, f_r, \dots\}$  by means of the Gram-Schmidt process. Then, it is important to remark that the linear manifolds generated by  $\{u_0, u_1, \dots, u_R\}$  and  $\{f_0, f_1, \dots, f_R\}$  are identical, *i.e.*,

$$\mathcal{L}(u_0, u_1, \dots, u_R) = \mathcal{L}(f_0, f_1, \dots, f_R) \tag{5.9}$$

for any  $R \in \mathbb{N}$ . Consequently, (5.8) is equivalent to

$$\sum_{i=1}^n q_i \langle \partial V_i, u_r \rangle = 0 \tag{5.10}$$

$r = 0, 1, 2, \dots, S$ , in the sense that both systems lead to the same set of solutions. Furthermore, the equivalence between both systems still holds if  $S$  is replaced by any  $R \in \mathbb{N}$ .

Consider the matrices of Fourier coefficients

$$\begin{pmatrix} \langle \partial V_1, f_0 \rangle, \langle \partial V_2, f_0 \rangle, \dots, \langle \partial V_n, f_0 \rangle \\ \langle \partial V_1, f_1 \rangle, \langle \partial V_2, f_1 \rangle, \dots, \langle \partial V_n, f_1 \rangle \\ \dots\dots\dots \\ \langle \partial V_1, f_S \rangle, \langle \partial V_2, f_S \rangle, \dots, \langle \partial V_n, f_S \rangle \end{pmatrix}$$

and

$$\begin{pmatrix} \langle \partial V_1, f_0 \rangle, \langle \partial V_2, f_0 \rangle, \dots, \langle \partial V_n, f_0 \rangle \\ \langle \partial V_1, f_1 \rangle, \langle \partial V_2, f_1 \rangle, \dots, \langle \partial V_n, f_1 \rangle \\ \dots\dots\dots \\ \langle \partial V_1, f_S \rangle, \langle \partial V_2, f_S \rangle, \dots, \langle \partial V_n, f_S \rangle \\ \langle \partial V_1, f_{S+1} \rangle, \langle \partial V_2, f_{S+1} \rangle, \dots, \langle \partial V_n, f_{S+1} \rangle \\ \dots\dots\dots \end{pmatrix},$$

where  $\mathcal{M}_\infty$  has an infinite number of rows. Clearly, their identical

$$R(\mathcal{M}_S) = R(\mathcal{M}_\infty) \tag{5.11}$$

ranges reflect the number of linearly independent rows, and cannot be larger than the number  $n$  of available assets. Furthermore, they provide the dimension of the space  $I_0$ , as well as the range  $R(\{\partial V_1, \partial V_2, \dots, \partial V_n\})$  of the family  $\{\partial V_1, \partial V_2, \dots, \partial V_n\} \subset H$ .

**Theorem 6** *The dimension of  $I_0$ , subspace of immunized portfolios, is given by*

$$Dim(I_0) = n - R(\mathcal{M}_S) = n - R(\mathcal{M}_\infty) = n - R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}).$$

*In particular,  $I_0$  reduces to zero if and only if*

$$R(\mathcal{M}_S) = R(\mathcal{M}_\infty) = R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}) = n.$$

**Proof.** The latter lemma shows that  $Dim(I_0) = n - R(\mathcal{M}_S) = n - R(\mathcal{M}_\infty)$ , so it only remains to prove that  $R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}) = R(\mathcal{M}_S)$ . Let be

$$m = R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}) \tag{6.12}$$

and, to make the notation easier, suppose that

$$\{\partial V_1, \partial V_2, \dots, \partial V_m\}$$

are independent and

$$\{\partial V_{m+1}, \partial V_{m+2}, \dots, \partial V_n\}$$

linearly depend on them. Thus, Columns  $(m+1)^h, (m+2)^h, \dots, n^h$  of  $\mathcal{M}_S$  also depend on Columns  $1^s, 2^d, \dots, m^h$ , from whence

$$R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}) \geq R(\mathcal{M}_S),$$

On the other hand, if

$$R(\{\partial V_1, \partial V_2, \dots, \partial V_n\}) > R(\mathcal{M}_S) = p,$$

then suppose that Columns  $(p+1)^h, \dots, m^h, \dots, n^h$  of  $\mathcal{M}_c$  depend on Columns  $1^s, 2^d, \dots, p^h$ . Then, (5.11) shows that Columns  $(p+1)^h, \dots, m^h, \dots, n^h$  of  $\mathcal{M}_r$  depend on Columns  $1^s, 2^d, \dots, p^h$  of  $\mathcal{M}_r$ , for every  $r \geq S$ , and with identical coordinates that do not depend on  $r \in \mathbb{N}$ . Then, (1.2) and (1.3) apply, and show that

$$\{\partial V_{p+1}, \partial V_{p+2}, \dots, \partial V_m\}$$

depend on

$$\{\partial V_1, \partial V_2, \dots, \partial V_p\},$$

which contradicts (6.12).



### 3. GENERAL HEDGING CRITERIA

Let us denote by  $Q \subset \mathfrak{R}^n$  the subset of feasible strategies.  $Q$  represents those portfolios satisfying several constraints that the investor (or the insurance company) must respect. For instance, if there are no restrictions, then

$$Q = \mathfrak{R}^n.$$

If we deal with the *CIP*, and the investor has to compose self-financing portfolios, as in Uberti (1997) and Hürlimann (2002), then

$$Q = \{q \in \mathfrak{R}^n ; \sum_{i=1}^n p_i q_i = 0\}.$$

If liabilities are fixed and can not be altered by the investor, as in the papers above, then we can consider that they generate the  $n^{\text{th}}$  – bond, and

$$Q = \{q \in \mathfrak{R}^n ; q_n = -1\}.$$

If short-sales are not allowed, and, the capital to invest equals  $C \in \mathfrak{R}$ , as in Fong and Vasicek (1984), Bierwag *et al.* (1993), or Balbás and Ibáñez (1998), amongst others, then

$$Q = \{q \in \mathfrak{R}^n ; \sum_{i=1}^n p_i q_i = C \text{ and } q_i \geq 0, i = 1, 2, \dots, n\},$$

Let us leave the *CIP* and return to the general case. Once the feasible set  $Q$  has been fixed, we will denote

$$Q_r = \{q \in Q; q \text{ is immunized versus } f_0, f_1, \dots, f_r\}$$

for  $r = 0, 1, 2, \dots$ , and

$$Q_\infty = \{q \in Q; q \text{ is immunized}\} = Q \cap I_0.$$

Lemma 5 shows that  $Q_\infty = Q_S$  for every  $Q \subset \mathfrak{R}^n$ .

Let us focus on the *CIP*. Literature usually provides hedging strategies by drawing on two general principles: Firstly, agents must choose an immunized portfolio if it exists (for instance, Fisher and Weil (1971) consider additive shifts and therefore recommend to select a duration matching strategy and, analogously, Chambers *et al.* (1988) characterize those portfolios immunized against  $m$  – degree polynomials,  $m < n$ , and recommend to adjust a duration vector). Secondly, if there are no immunized portfolios, then authors recommend to immunize against the most important factors of risk, and to minimize the “residual risk”. Significant references are, amongst others, Fong and Vasicek (1984), where it is recommended to minimize the  $M$  – squared measure among duration matching strategies, because there is no perfect immunization against continuously differentiable shocks if short-sales are forbidden, or Balbás *et al.* (2002)  $b$ , where short-selling restrictions are imposed too and several risk measures are proposed, minimized and empirically tested.

Going back to the general case and leaving the *CIP*, but following the literature, if possible, we also recommend to select immunized portfolios:

**First general hedging criterion (Criterion 1).** If  $Q_\infty \neq \emptyset$ , choose a portfolio  $q \in Q_\infty$ .

However, if (as usual)  $0 \notin Q$ , then  $Q_\infty \neq \emptyset$  needs  $I_0 \setminus \{0\}$  to be non-empty, and Theorem 6 implies that

$$\{\partial V_1, \partial V_2, \dots, \partial V_n\} \subset H$$

has to be linearly dependent. Standard literature deals with models such that  $Dim(H) < n$  and, accordingly, the dependence of the family above is guaranteed. In our setting  $Dim(H) = \infty$ , and this dependence will hardly hold. Furthermore, under the *CIP* case, it is known that the parameters of the problem (prices, *TSIR*, durations, etc.) are dynamic, and agents must frequently rebalance their strategies in order to get adequate hedging. So, even if Criterion 1 is attainable, it may become infeasible some periods later.

A special important case, that makes the dependence of  $\{\partial V_1, \partial V_2, \dots, \partial V_n\}$  stable as time goes back, arises if at least one security in  $\{B_1, B_2, \dots, B_n\}$  may be replicated by using the remainder ones. It is not usual but, for instance, it might hold if we incorporate forwards and futures in the analysis.

Since Criterion 1 may fail when dealing with infinite dimensions, we introduce:

**Second general criterion (Criterion 2).** Assume that  $Q \cap I_0 = \emptyset$ . Compute the maximum  $R$  with  $Q_R \neq \emptyset$ , and solve the optimization problem

$$\text{Min} \sum_{r=R+1}^{\infty} \langle \partial V_q, f_r \rangle^2 \{q \in Q_R\} \tag{6.13}$$

Since  $Q_S = \emptyset$ , the existence of  $R$  is guaranteed. The constraint  $q \in Q_R$  ensures that we are immunizing against the empirically most important factors of risk  $\{f_0, f_1, \dots, f_R\}$ . Besides, (1.2) shows that

$$\partial V_q = \sum_{r=0}^R \langle \partial V_q, f_r \rangle f_r + \sum_{r=R+1}^{\infty} \langle \partial V_q, f_r \rangle f_r = \pi_{\mathcal{L}}(\partial V_q) + \pi_{\mathcal{L}^\perp}(\partial V_q),$$

and  $\partial V_q$  may be obtained by adding its projection on  $\mathcal{L}(f_0, f_1, \dots, f_R)$ , linear manifold generated by factors  $0, 1^{st}, 2^{nd}, \dots, R^{th}$ , and the projection on its orthogonal  $\mathcal{L}(f_0, f_1, \dots, f_R)^\perp$ . The constraint  $q \in Q_R$  makes the first projection vanish, and the objective function of (6.13), along with Parsevals equality and Proposition 1, point out that we are following a minimax principle, as Bowden (1997), or Barber and Copper (1998).

**Theorem 7** Problem (13) and Problem

$$\text{Min}_{q \in Q_R} \left( \text{Max}_{\{u \in H, \|u\| \leq 1\}} \left| D_u V_q \right|^2 \right) \tag{7.14}$$

are equivalent, in the sense that they have the same solution  $\tilde{q} \in Q_R$  and the same optimal value  $\|\partial V_{\tilde{q}}\|^2$ . Moreover, for  $q = \tilde{q}$ , one has that the objective function of (14) is attained at

$$u = \frac{\partial V_{\tilde{q}}}{\|\partial V_{\tilde{q}}\|}, \quad (7.15)$$

and this shift will be called worst shock of (6.13) or (7.14).

**Proof.** Suppose that  $q \in Q_R$ . Since  $\partial V_q \neq 0$  ( $Q_\infty = Q \cap I_0 = \emptyset$ , so  $q \notin I_0$  and Proposition 1 applies), Proposition 1 shows that the objective function of (7.14) is attained at  $u = \frac{\partial V_q}{\|\partial V_q\|}$ , and achieves the value  $\|\partial V_q\|^2$ . Besides, Parsevals equality points out that

$$\|\partial V_q\|^2 = \sum_{r=0}^{\infty} \langle \partial V_q, f_r \rangle^2 = \sum_{r=R+1}^{\infty} \langle \partial V_q, f_r \rangle^2$$

because  $q \in Q_R$  and, therefore,  $\langle \partial V_q, f_r \rangle = 0$  if  $r \leq R$ .

Theorem 7 illustrates several analogies and differences between our proposal and the hedging criterion introduced in Bowden (1997), or, in Barber and Copper (1998). These authors deal with the CIP and solve

$$\text{Min}_{q \in Q} \left( \text{Max}_{\{u \in H, \|u\| \leq 1\}} |D_u V_q|^2 \right), \quad (7.16)$$

analogous to (7.14), with the constraint  $q \in Q$ , rather than  $q \in Q_R$ . Accordingly (see Proposition 1), they reach a better result, in the sense that their optimal sensitivity is lower than ours, *i.e.*,

$$\|\partial V_{q^*}\| \leq \|\partial V_{\tilde{q}}\|,$$

if  $q^*$  solves the problem of Barber and Copper (1998), and  $\tilde{q}$  solves (6.13) or (7.14). Nevertheless, a possible advantage of our analysis arises if we compare the worst (most negative) scenarios. As said above, Problem (7.14) maximizes on  $\frac{\partial V_{\tilde{q}}}{\|\partial V_{\tilde{q}}\|}$ , and this shift is orthogonal to  $\mathcal{L}(f_0, f_1, \dots, f_R, f_R)$ . Thus,

if we have appropriately ranked the risk factors according to the empirical evidence, then our worst scenario may be “unrealistic” and improbable. On the contrary, the worst scenario  $\frac{\partial V_{q^*}}{\|\partial V_{q^*}\|}$  of papers

above (Direction  $X$ , in the terminology of Bowden (1997)) could be very probable, and  $q^*$  could be unprotected against the most usual shocks. Some examples provided by Barber and Copper (1998) illustrate that their optimal strategy is not necessarily immunized against additive shifts, which is a real drawback if we observe how important these shocks are in practice (see Litterman and Scheinkman (1991), Bierwag *et al.* (1993), Chance and Jordan (1996) or Balbás *et al.* (2002) *b*). So, though Criterion 2 generates a larger minimax value, it may perform much better since it immunizes with regard to standard shocks, and the minimax value only hedges the residual risk.

In some sense, our proposal retrieves some “consensus” between the general criterion applied in literature (immunize against probable shocks and protect against residual risk) and the minimax criterion of Barber and Copper (1998). The consensus has been achieved due to the ranking of risks we have previously chosen.

Since Criterion 2 does not incorporate any distinction among the ranked factors  $(f_r)_{r=R+1}^\infty$ , we also propose:

**Third general criterion (Criterion 3).** Assume that  $Q \cap I_0 = \emptyset$ . Compute the maximum  $R$  with  $Q_R \neq \emptyset$ , and solve the optimization problem

$$\text{Min } \prec \partial V_q, f_{R+1} \succ^2 \{q \in Q_R. \tag{7.17 a}$$

Equality  $Q_S = \emptyset$  again guarantees the existence of  $R$ . The major difference between Criterion 2 and Criterion 3 lies in the measurement of the residual risk. In this new case, we abandon the minimax principle and consider the first factor making it impossible to immunize. Thus, we minimize the absolute value of the first non-null coordinate of  $\partial V_q$ , rather than its norm. Criterion 3 overcomes Criterion 2 in the sense that it hedges as much as possible against the most important factor  $f_{R+1}$ , but the global sensitivity (i.e.,  $\|\partial V_q\|$ ) increases if we apply Criterion 3.

Focusing on the *CIP*, one has that Criterion 3 generalizes the seminal proposal of Fong and Vasicek (1984). They considered differentiable shifts on the *TSIR*, and showed that duration matching portfolios guarantee an amount of money bounded from below by the shock slope (by  $\prec \partial V_q, f_1 \succ^2$ , in our context). Furthermore, their result was significantly extended in the important contributions of Shiu (1987) and (1990), as well as in the papers of Montrucchio and Peccati (1991), Uberti (1997) and Hürlimann (2002), where it is proved that the risk level with regard to the shock slope ( $\prec \partial V_q, f_1 \succ^2$ , in our context) also bounds capital losses for quite general shocks, far from differentiable.

Theorem 8 below provides an additional reason to justify Criteria 2 and 3, as well as allows us to establish general lower bounds for capital losses in our general setting. In some sense, we complement those inequalities of Fong and Vasicek (1984), Shiu (1987, 1990) and their extensions. It is also possible to prove that the lower bounds introduced in Nawhalka and Chambers (1996) and Balbás *et al.* (2002) *b* are particular cases of this theorem.

**Theorem 8** Let  $q \in \mathfrak{R}^n$ . Assume that  $V_q : H \rightarrow \mathfrak{R}$  is a convex function<sup>8</sup>. Denote by  $\pi_{\mathcal{L}^\perp}$  the projection from  $H$  on the orthogonal subspace of the linear manifold generated by  $\{f_0, f_1, \dots, f_R\}$ .

If  $q$  solves (6.13), and  $\mathfrak{R}_2$  is the achieved risk level (the optimal objective value), then

$$V_q(h) - V_q(0) = \sqrt{\mathfrak{R}_2} \|\pi_{\mathcal{L}^\perp}(h)\| \tag{7.17 a}$$

holds for every  $h \in H$ .<sup>9</sup> Additionally, if the price of  $q$  is positive, then capital losses in percentage verify

<sup>8</sup> For instance, if  $V_i$  is convex  $i=1,2,\dots,n$  and  $q$  does not contains short-sales ( $q_i \geq 0, i=1,2,\dots,n$ ).

<sup>9</sup> Since,  
 $\|h\|^2 = \|\pi_{\mathcal{L}}(h)\|^2 + \|\pi_{\mathcal{L}^\perp}(h)\|^2$   
 the inequality above may be written as  
 $V_q(h) - V_q(0) \geq -\sqrt{\mathfrak{R}_2} (\|h\|^2 - \|\pi_{\mathcal{L}^\perp}(h)\|^2)$   
 $= -\sqrt{\mathfrak{R}_2} (\sum_{r=R+1}^n h_r^2)$   
 for every  $h \in H$ . In particular,  
 $V_q(h) - V_q(0) \geq -\sqrt{\mathfrak{R}_2} \|h\|$   
 for every  $h \in H$ .

$$\frac{V_q(h)}{V_q(0)} \geq 1 - \frac{\sqrt{\mathcal{R}_2}}{V_q(0)} \|\pi_{\mathcal{L}^T}(h)\| \quad (7.17 \text{ b})$$

for every  $h \in H$ .

If  $q$  solves (7.17a), and  $\mathcal{R}_3$  is the achieved risk level, then

$$V_q(h) - V_q(0) \geq -\sqrt{\mathcal{R}_3} \|h_{R+1}\| \quad (7.17 \text{ c})$$

holds for every  $h \in \mathcal{L}\mathcal{L}(f_0, f_1, \dots, f_R, f_{R+1})$ . Moreover, if the price of  $q$  is positive, then

$$\frac{V_q(h)}{V_q(0)} \geq 1 - \frac{\sqrt{\mathcal{R}_3}}{V_q(0)} \|h_{R+1}\| \quad (7.17 \text{ d})$$

for every  $h \in \mathcal{L}(f_0, f_1, \dots, f_R, f_{R+1})$ .

**Proof.** It is known that any convex function is larger than its tangent hyperplane (see Luenberger (1969)). Therefore,

$$\begin{aligned} V_q(h) - V_q(0) &\geq \langle \partial V_q, h \rangle \\ &= \langle \partial V_q, \pi_{\mathcal{L}}(h) + \pi_{\mathcal{L}^T}(h) \rangle = \langle \partial V_q, \pi_{\mathcal{L}^T}(h) \rangle \end{aligned} \quad (8.5)$$

for every  $h \in H$ . Then, the Cauchy-Schwartz inequality leads to

$$V_q(h) - V_q(0) \geq - \left| \langle \partial V_q, \pi_{\mathcal{L}^T}(h) \rangle \right| \geq - \|\pi_{\mathcal{L}^T}(h)\| \|\partial V_q\|$$

for every  $h \in H$ , whence (7.17a) and (7.17b) trivially follow, because  $\|\partial V_q\| = \sqrt{\mathcal{R}_2}$ .

Suppose now that  $h \in \mathcal{L}(f_0, f_1, \dots, f_R, f_{R+1})$ , and  $q \in \mathcal{Q}_R$ . Then, (8.5) and (1.4) show that

$$V_q(h) - V_q(0) \geq (\partial V_q)_{R+1} h_{R+1} = \langle \partial V_q, f_{R+1} \rangle h_{R+1},$$

whence the conclusion immediately follows.

**Remark 1** Recent literature has proposed to hedge the credit risk by using risk measures (see for instance Vlaar (2000), Andersson (2001) or Rockafellar and Uryasev (2002)). The analysis of possible relationships between this new approach and the classical linked-duration methods is beyond our scope, but it is worth to point out that Theorem 8 may be useful. For instance, suppose that the set  $H$  of feasible shocks is endowed with an appropriate probability measure, in such a way that  $\|h\|$  and  $V_q(h)$  become continuous random variables for every  $q \in \mathfrak{R}^n$ . Fix  $p \in (0, 1)$ , and define the Value at Risk of  $q \in \mathfrak{R}^n$  according to

$$VaR(q) = \alpha \Leftrightarrow \text{prob}\{V_q(h) - V_0(h) \leq -\alpha\} = p.$$

Suppose that  $q$  solves (6.13) and, under convexity assumptions, Expression (7.17a) clearly shows that

$$prob \left\{ \left\| \pi_{\mathcal{L}^T}(h) \right\| = \left( \sum_{r=R+1}^{\infty} h_r^2 \right) \geq \frac{\alpha}{\sqrt{\mathcal{R}_2}} \right\} \geq p.$$

Hence,

$$prob \left\{ \left( \sum_{r=R+1}^{\infty} h_r^2 \right) \geq \frac{VaR(q)}{\sqrt{\mathcal{R}_2}} \right\} \geq p,$$

According to the distribution of  $\left( \sum_{r=R+1}^{\infty} h_r^2 \right)^{1/2}$ , take  $\beta$ , such that

$$prob \left\{ \left( \sum_{r=R+1}^{\infty} h_r^2 \right)^{1/2} \geq \beta \right\} = p,$$

and we have that

$$VaR(q) \leq \beta \sqrt{\mathcal{R}_2}.$$

**Remark 2** *It is worth to point out that expressions above are also interesting if the convexity fails. In fact, although they do not provide upper bonds for capital losses, they do approximate the portfolio price after the most negative scenarios.*

It is easy to provide counter-examples illustrating the absence of any kind of relationships between the solutions of (6.13) and (7.17a). Hence, we also propose:

**Last general criterion (Criterion 4).** Assume that  $Q \cap I_0 = \emptyset$ . Compute the maximum  $R$  with  $Q_R \neq \emptyset$ , and solve the optimization problem

$$Min \sum_{r=R+1}^{\infty} \langle \partial V_q, f_r \rangle^2 \begin{cases} q \in Q_R \\ |\langle \partial V_q, f_r \rangle| \leq L_r \quad r = R+1, R+2, \dots \end{cases} \quad (8.6)$$

Clearly, Criterion 4 becomes Criterion 2 if  $L_r = \infty$ ,  $r = R+1, R+2, \dots$

The following result, whose proof is parallel to the proof of Theorem 7 and therefore it is omitted, guarantees that Criterion 4 also obeys to minimax-like principles.

**Theorem 9** *Problem (8.6) and Problem*

$$Min_{\{q \in Q_R, |\langle \partial V_q, f_r \rangle| \leq L_r, r=R+1, R+2, \dots\}} \left( Max_{\{u \in H, \|u\| \leq 1\}} \left| D_u V_q \right|^2 \right) \quad (9.7)$$

are equivalent, in the sense that they have the same solution  $q \in Q_R$ , and the same optimal value

$\left\| \partial V_q \right\|^2$ . Moreover, for  $q = \tilde{q}$ , one has that the objective function of (9.7) is attained at

$$u = \frac{\partial V_{\tilde{q}}}{\left\| \partial V_{\tilde{q}} \right\|}, \quad (9.8)$$

and this shift will be called worst shock of (8.6) or (9.7).

Notice that (7.17a) and (7.17b) also apply if  $q$  solves (8.6) and  $\mathcal{R}_2$  is substituted by the obvious optimal value  $\mathcal{R}_4$ .

Criterion 4 may be also very useful in practice because it incorporates the positive properties of Criteria 2 and 3. Indeed, suppose that we are very interested in controlling the whole risk level  $\|\partial V_q\|$ , but the solution of (6.13) reflects a high value of

$$|\langle \partial V_q, f_{R+1} \rangle|$$

first risk-level making immunization infeasible.<sup>10</sup> Then Criterion 4 permits us to retrieve the mini-max rule (the minimization of  $\partial V_q$ ) by choosing acceptable levels  $L_r \leq \infty$  for the partial risks

$$|\langle \partial V_q, f_r \rangle|$$

$$r = R + 1, R + 2, \dots^{11}$$

## 4. COMPUTING HEDGING PORTFOLIOS IN PRACTICE

### 4.1 Immunizing default-free bond portfolios

Firstly, we will focus on the classical immunization problem *CIP*. So, all the available securities will be credit-risk-free bonds (Footnote 3 applies here),  $H = L^2[0, T]$  and  $H \ni h_0$ , where  $h_0$  is denoting the initial *TSIR*.

Consider the set

$$0 < t_1 < t_2 < \dots < t_{k-1} < t_k = T \in [0, T],$$

10 In some sense, under convexity assumptions (7.17c) and (7.17d) would be generating “bad bounds” for possible capital losses.

11 There exists another way to incorporate the advantages of both Criteria 2 and 3. We could solve the vector optimization problem

$$\text{Min} \left( \sum_{r=R+1}^{\infty} |\langle \partial V_q, f_r \rangle|^2, |\langle \partial V_q, f_{R+1} \rangle|^2 \right) \{q \in Q_R\}$$

This may be done by using balance point techniques. In such a case, we must compute  $\mathcal{R}_2$  and  $\mathcal{R}_3$ , optimal levels of Criteria 2 and 3, choose the direction  $(1, d)$ , indicating the ratio  $d > 0$  of losses in the second objective per unit lost in the first one, and solve the scalar problem

$$\text{Min } \lambda \left\{ \begin{array}{l} q \in Q_R \\ \|\partial V_q\|^2 - \lambda \leq \mathcal{R}_2 \\ |\langle \partial V_q, f_{R+1} \rangle|^2 - d\lambda \leq \mathcal{R}_3 \end{array} \right.$$

with  $(\lambda, q)$  being the decision variable. If  $(\lambda, q)$  solves the latter problem, then

$$\mathcal{R}_2 + \lambda,$$

and

$$\mathcal{R}_3 + d\lambda$$

are the reached values of both objectives. Moreover, if  $V_q$  is convex (and  $V_q(0) > 0$ , if necessary), then the lower bounds of Theorem 8 apply as long as one replaces  $\mathcal{R}_2$  and  $\mathcal{R}_3$  by the values above  $\mathcal{R}_2 + \lambda$  and  $\mathcal{R}_3 + d\lambda$  (see Balbás *et al.* (2002) *a* for further details on balance points).

indicating those dates when  $B_j$  pays the (maybe null) coupon  $c_{i,j}$ ,  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, n$ . Then, for every portfolio  $q = (q_1, q_2, \dots, q_n) \in \mathfrak{R}^n$ , one can compute the (maybe non-positive) coupon

$$c_i = \sum_{j=1}^n q_j c_{i,j} \tag{9.9}$$

paid at  $t_i$ ,  $i = 1, 2, \dots, k$ . It is known that, after a shock  $h \in L^2[0, T]$ , one has

$$V_q(h) = \sum_{i=1}^k c_i \left[ \exp\left(-\int_0^{t_i} (h_0(u) + h(u)) du\right) \right],$$

which becomes

$$V_q(h) = \sum_{i=1}^k \tilde{c}_i \left[ \exp\left(-\int_0^{t_i} h(u) du\right) \right], \tag{9.10}$$

if

$$\tilde{c}_i = c_i \left[ \exp\left(-\int_0^{t_i} h_0(u) du\right) \right] \tag{9.11}$$

represents the present value of  $c_i$ ,  $i = 1, 2, \dots, k$ . Moreover, Barber and Copper (1998) prove that  $\partial V_q \in L^2[0, T]$  is the piecewise constant and bounded function<sup>12</sup>

$$\partial V_q(t) = -\sum_{t_i \geq t} \tilde{c}_i \tag{9.12}$$

for each  $t \in [0, T]$ . Consequently, the existence of immunized portfolios verifying  $\partial V_q = 0$  will hold as long as  $\tilde{c}_i = 0$ ,  $i = 1, 2, \dots, k$ , which, according to (9.11), leads to the following result:

**Theorem 10** *Portfolio  $q$  is immunized if and only if its coupons vanish.*

It immediately follows that Criterion 1 does not apply unless the available assets  $B_1, B_2, \dots, B_n$  are not independent. If they are independent, or their replicas of “zero” are not feasible (do not belong to the set  $Q$ ), then Barber and Copper (1998) suggest to chose that portfolio  $q$  solving (7.16). Thus, according to (9.12), managers must minimize

$$\|\partial V_q\|^2 = \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right)^2 (t_i - t_{i-1}) \right]$$

among the feasible portfolios  $q \in Q$ , where  $t_0 = 0$  represents the current date (see the reference above for further details).

As already said, the simple worst shock (9.12) could be very unrealistic, and we would be hedging against strange shifts, losing protection against far more important (for instance, additive) shocks.

As a consequence, it may be more suitable to apply our Criteria 2, 3 or 4. If so, we have several alternatives when ranking the factors, although we will concentrate our analysis on two quite general situations justified by the empirical evidence. So, let us consider, respectively, factors such

<sup>12</sup> From now on we will merely say “simple” function.



as “level”, “slope”, “curvature”, etc. (see Litterman and Scheinkman (1991)), or, as in the next subsection, factors linked to spot rates (see Elton *et al.* (1990)).

It is known that the set of polynomials  $\{u_\alpha(t) = t^\alpha : \alpha = 0, 1, 2, \dots\}$  is a Schauder basis of  $L^2[0, T]$ , so one can build the orthonormal basis  $\{f_\alpha\}_{\alpha=0}^\infty$  by applying the Gram-Schmidt process. According to (10), given  $R \in \mathbb{N}$ , the portfolio  $q$  is immunized against  $R$  – degree polynomials ( *i.e.*, (5.7) holds for  $r = 0, 1, \dots, R$  ) if and only if

$$\int_0^T t^\alpha \partial V_q(t) dt = 0,$$

$\alpha = 0, 1, \dots, R$ . Therefore,

$$0 = \sum_{\beta=1}^k \partial V_q(t_\beta) \frac{t_\beta^{\alpha+1} - t_\beta^\alpha}{\alpha + 1}$$

whence

$$\begin{aligned} 0 &= \frac{1}{\alpha + 1} \left( \sum_{\beta=1}^k (t_\beta^{\alpha+1} - t_\beta^\alpha) \left( \sum_{i=\beta}^k \tilde{c}_i \right) \right) \\ &= \frac{1}{\alpha + 1} \sum_{i=1}^k \tilde{c}_i \left( \sum_{\beta=1}^i (t_\beta^{\alpha+1} - t_\beta^\alpha) \right) \\ &= \frac{1}{\alpha + 1} \sum_{i=1}^k \tilde{c}_i t_i^{\alpha+1} \end{aligned}$$

$\alpha = 0, 1, \dots, R$ , which trivially leads to

$$\sum_{i=1}^k \tilde{c}_i t_i^{\alpha+1} = 0 \tag{10.13}$$

$\alpha = 0, 1, \dots, R$ . This condition was also obtained in Chambers *et al.* (1988) by using different arguments.

It is obvious that (10.13) holds if and only if assets and liabilities associated with  $q$  have similar  $\alpha$  – duration (or duration of order  $\alpha$ ), where the  $\alpha$  – duration of an arbitrary portfolio  $q^*$ , with discounted coupons  $\tilde{c}_1^*, \tilde{c}_2^*, \dots, \tilde{c}_k^*$  and price  $\sum_{i=1}^k \tilde{c}_i^* \neq 0$ , is given by

$$\frac{\sum_{i=1}^k \tilde{c}_i^* t_i^{\alpha+1}}{\sum_{i=1}^k \tilde{c}_i^*}. \tag{10.14}$$

Let us assume that  $B_1, B_2, \dots, B_n$  are independent and (as usual)  $0 \notin Q$ . According to Proposition 2 and Theorem 10, Expression (13) cannot hold for every  $\alpha \in \mathbb{N}$ . Besides, (5.0) and Lemma 5 ensure the existence of a highest  $R$ , such that  $q \in Q$ , along with (10.13)  $\alpha = 0, 1, \dots, R$ , generate a non-void set. Moreover, this set contains those portfolios immunized against  $R$  – degree polynomials. Criteria 2, 3 and 4 become, respectively (see (9.12)),

$$\text{Min} \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right)^2 (t_i - t_{i-1}) \right] \left\{ \begin{array}{l} q \in Q \\ \sum_{i=1}^k \tilde{c}_i t_i^{\alpha+1} = 0 \quad \alpha = 0, 1, \dots, R \end{array} \right.$$

$$\text{Min} \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right)^2 (t_i - t_{i-1}) \right] \left\{ \begin{array}{l} q \in Q \\ \sum_{i=1}^k \tilde{c}_i t_i^{\alpha+1} = 0 \quad \alpha = 0, 1, \dots, R' \end{array} \right.$$

and

$$\text{Min} \left( \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right) \int_{t_{i-1}}^{t_i} f_{R+1}(t) dt \right]^2 \right) \left\{ \begin{array}{l} q \in Q \\ \sum_{i=1}^k \tilde{c}_i t_i^{\alpha+1} = 0 \quad \alpha = 0, 1, \dots, R \end{array} \right.$$

Notice that there are no practical difficulties to solve the problems above. In fact, on the one hand, the family of orthonormal polynomials is easily computed by bearing in mind that the degree of every  $f_r$  equals  $r$ , applying a simple induction, and imposing  $\int_0^T f_r(t)^2 dt = 1$  and,  $r, s = 0, 1, 2, \dots, s \neq r$  (in order to prevent the computation of the whole orthonormal basis, it may be convenient to impose  $L_s = \infty$  for  $S$  large enough). On the other hand, taking into account (9.9) and (9.11), and keeping aside the constraint  $q \in Q$ , the three problems above are quadratic in the  $q$  – variable, and therefore they are easily solved by those techniques presented, for instance, in Luenberger (1969).<sup>13</sup> Let us finally remark that, in practice, the set  $Q$  is very often given by means of linear or quadratic restrictions (see the beginning of Section 3).

#### 4.2 Immunizing against ranked spot rates

Expression (9.12) implies that the gradient of any feasible portfolio will always be a simple function. Consequently, the worst shocks associated with Criteria 2 and 4, as well as the Direction  $X$  of Bowden (1997), will be simple functions. Thus, it may be appropriate to immunize against aN orthonormal set  $\{f_r\}_{r=0}^\infty$ , such that  $f_r$  is simple for every  $r \in \mathbb{N}$ . Furthermore, this sort of orthonormal factors may be adequate if the  $TSIR$  is represented by (non orthonormal) factors that are spot rates. Indeed, as pointed out by Elton *et al.* (1990), spot rates work well when representing the whole  $TSIR$  by a small number of factors. In addition, Navarro and Nave (1997) showed that, for immunization or hedging purposes, the methodology of Elton *et al.* works well from an empirical viewpoint.

Throughout this subsection, we will consider the same framework as in the previous one, in such a way that all the available securities will be credit-risk-free bonds (Footnote 3 applies here),  $H = L^2[0, T]$  and  $h_0 \in H$  represent the initial  $TSIR$ . Symbols  $q$ ,  $B_j$ ,  $t_i$ ,  $c_i$ ,  $\tilde{c}_i$ ,  $c_{ij}$  and  $\tilde{c}_{ij}$  have the same interpretation as they had.

Suppose that the methodology of Elton *et al.* (1990) permits us to rank the spot rates associated with the set of dates  $\{t_1, t_2, \dots, t_k\}$ . Suppose also that  $t_{i_0}$  is the most important date. Define the first factor<sup>14</sup>

<sup>13</sup> Notice that the second problem can also be simplified if short-sales are not allowed and  $f_{R+1} \geq 0$ . In fact, in such a case the objective function may be replaced by

$$\sum_{j=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right) \int_{t_{i-1}}^{t_i} f_{R+1}(t) dt \right]^2,$$

that becomes linear in the  $q$  – variable.

<sup>14</sup> Recall that  $t_0 = 0$ . Throughout this section, take also  $t_{k+1} = \infty$  and  $\frac{1}{\infty} = 0$ , if necessary.

$$u_0(t) = \begin{cases} 0 & t \leq t_{i_0-1} \\ 1 & t_{i_0-1} < t \leq t_{i_0} \\ -\frac{t_{i_0} - t_{i_0-1}}{t_{i_0+1} - t_{i_0}} & t_{i_0} < t \leq t_{i_0+1} \\ 0 & t > t_{i_0+1} \end{cases}. \quad (10.15)$$

Then, it is easy to show that any shock on the *TSIR* in the direction of  $u_0$  leads to the new term structure  $h_0 + \lambda u_0$ , and only modifies the spot rate associated with  $t_{i_0}$ . Thus, following the methodology above, one can build the sequence of ranked factors  $\{u_0, u_1, \dots, u_{k-1}\}$  that contains all the relevant spot rates  $\{t_{i_0}, t_{i_1}, \dots, t_{i_{k-1}}\}$ .<sup>15</sup> Clearly,  $u_\alpha$  is given by (10.15) if  $i_0$  is substituted by  $i_\alpha$ ,  $\alpha = 0, 1, \dots, k-1$ .

The Gram-Schmidt process allows us to introduce the orthonormal set  $\{f_0, f_1, \dots, f_{k-1}\}$ .<sup>16</sup>

Let  $q \in \mathfrak{R}^n$  and  $R = 0, 1, 2, \dots, k-1$ . According to (5.9) and (5.10), Condition  $q \in \mathcal{Q}_R$  implies that

$$\begin{aligned} 0 &= \int_0^T u_\alpha(t) \partial V_q(t) dt \\ &= \int_{t_{i_\alpha-1}}^{t_{i_\alpha}} \partial V_q(t) dt - \frac{t_{i_\alpha} - t_{i_\alpha-1}}{t_{i_\alpha+1} - t_{i_\alpha}} \int_{t_{i_\alpha}}^{t_{i_\alpha+1}} \partial V_q(t) dt. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= (t_{i_\alpha} - t_{i_\alpha-1}) \left( \sum_{\beta=i_\alpha}^k \tilde{c}_\beta \right) - \frac{t_{i_\alpha} - t_{i_\alpha-1}}{t_{i_\alpha+1} - t_{i_\alpha}} \left( \sum_{\beta=i_\alpha+1}^k \tilde{c}_\beta \right) (t_{i_\alpha+1} - t_{i_\alpha}) \\ &= (t_{i_\alpha} - t_{i_\alpha-1}) \tilde{c}_{i_\alpha} \end{aligned}$$

$\alpha = 0, 1, \dots, R$ . Consequently, we have:

**Theorem 11** Let  $q \in \mathfrak{R}^n$  and  $R = 0, 1, 2, \dots, k-1$ . Then,  $q \in \mathcal{Q}_R$  if, and only if,  $q \in \mathcal{Q}$ , and

$$c_{i_0} = c_{i_1} = \dots = c_{i_R} = 0.$$

Theorem above generalizes Theorem 10 if we take  $R = k-1$ , so once again we obtain that Criterion 1 does not apply for independent securities. In such a scenario, the remainder criteria need the computation of  $R$ , maximum number such that System

$$\begin{cases} q \in \mathcal{Q} \\ \sum_{j=1}^n c_{i_\alpha j} q_j = 0 \quad \alpha = 0, 1, \dots, R \end{cases}$$

<sup>15</sup> Actually,  $\{t_{i_0}, t_{i_1}, \dots, t_{i_{k-1}}\} = \{t_1, t_2, \dots, t_k\}$ , though dates have been probably written with different order.

<sup>16</sup> It is possible, but useless, to extend the system  $\{f_0, f_1, \dots, f_{k-1}\}$  to a basis of  $L^2[0, T]$ , since functions  $V_q$  have no sensitivity (or null sensitivity) with respect to  $f_r$  for  $r \geq k$ .

has non-void solution. Accordingly, Criteria 2, 3 and 4 become, respectively,

$$\begin{aligned} & \text{Min} \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right)^2 (t_i - t_{i-1}) \right] \left\{ \begin{array}{l} q \in Q \\ \sum_{j=1}^n c_{i_{\alpha j}} q_j = 0 \quad \alpha = 0, 1, \dots, R \end{array} \right. \\ & \text{Min} \left( \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right) \int_{t_{i-1}}^{t_i} f_{R+1}(t) dt \right]^2 \right) \left\{ \begin{array}{l} q \in Q \\ \sum_{j=1}^n c_{i_{\alpha j}} q_j = 0 \quad \alpha = 0, 1, \dots, R \end{array} \right. \end{aligned}$$

and

$$\text{Min} \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right)^2 (t_i - t_{i-1}) \right] \left\{ \begin{array}{l} q \in Q \\ \sum_{j=1}^n c_{i_{\alpha j}} q_j = 0 \quad \alpha = 0, 1, \dots, R \\ \left| \sum_{i=1}^k \left[ \left( \sum_{j=i}^k \tilde{c}_j \right) \int_{t_{i-1}}^{t_i} f_s(t) dt \right] \right| \leq L_s \quad s = i_{R+1}, i_{R+2}, \dots, i_{k-1} \end{array} \right.$$

As in the previous subsection, problems above are easily solved in practice.

The major difference between both possibilities, polynomials or spot rates, relies on the second constraint. In this second case, we do not adjust any duration vector, but make the most sensitive coupons vanish. Market conditions and the empirical evidence should provide the arguments to select between both approaches, although theoretical reasons could also be considered. For instance, investors could bear in mind the upper bounds of Theorem 8. Finally, let us mention that both possibilities (polynomials and spot rates) may be simultaneously combined in a single analysis.

### 4.3 Incorporating default risk

Many risk-adverse agents, insurers and pension funds are considering non default-free bonds when choosing their portfolios. The reasons seem to be clear, since the credit spread is becoming more and more significant in the market. So, for instance, if one tests the period 2000-2004, the European bond markets have been generating spreads close to 300 basic points for those companies with the highest rating, *i.e.*, the return associated with private bonds almost multiplies by two the return of public bonds. Furthermore, private bonds are showing small volatilities, much lower than those reflected by shares or derivatives, which makes them very suitable when composing hedge strategies.

Another reason (perhaps less important) that makes it rather convenient to consider the credit spread level is linked to the insurers risk or rating. Indeed, if the market imposed any risk premium, then some insurance companies should take it into account in order to evaluate short-sales, since discount factors might be different when pricing positive or negative cash flows.

The literature has recently treated the credit risk measurement and control by using "risk measures". For example, Vlaar (2000) uses the Value at Risk (*VaR*), while Andersson *et al.* (2001) and Rockafellar and Uryasev (2002) prefer the Conditional Value at Risk (*CVaR*). It is also possible to draw on the coherent measures of Artzner *et al.* (1999), or the convex measures of Föllmer and

Schied (2002). All of these measures provide an amount of money that the portfolio managers must add in order to protect their clients if the market evolution is quite negative.

Despite the comment above, it is worth to recall that the risk measurement in finance is very often related to sensitivities between financial variables. For instance, the risk of derivative portfolios is usually measured by the Greeks (see Ingersoll (1987) for further details). Moreover, since the interest-rate-risk is frequently measured by means of sensitivities (durations), it may be convenient to extend the analysis in order to capture the default-risk. The study of possible relationships between both approaches is beyond our current purposes, and left for future research.

Throughout this subsection  $B_1, B_2, \dots, B_r$  will be default-free bonds, while  $B_{r+1}, B_{r+2}, \dots, B_n$  will be bonds reflecting credit risk. We will use the notations of Subsections 4.1 and 4.2, so  $h_0$  will represent the initial *TSIR*. We will assume that all of the private bonds have similar rating and therefore reflect quite similar spread.<sup>17</sup> So, there exists  $h_0^* \in L^2[0, T]$ , indicating a “common spread” that applies for  $B_{r+1}, B_{r+2}, \dots, B_n$ . Finally, each private bond has a “less important specific spread”  $h_0^j \in L^2[0, T]$ ,  $j = r+1, r+2, \dots, n$ . Accordingly, the present value of  $c_{i,j}$  is given by

$$\tilde{c}_{i,j} = c_{i,j} e^{-\int_0^t h_0(u) du}$$

for  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, r$ , and

$$\tilde{c}_{i,j} = c_{i,j} e^{-\int_0^t (h_0^*(u) + h_0^j(u) + h_0(u)) du}$$

for  $i = 1, 2, \dots, k$  and  $j = r+1, \dots, n$ , and the price of a portfolio  $q = (q_1, q_2, \dots, q_n)$  is

$$\sum_{j=1}^n q_j \left( \sum_{i=1}^k \tilde{c}_{i,j} \right).$$

The Hilbert space will be

$$H = L^2[0, T] \times \left( L^2[0, T] \right)^{n-r} \times L^2[0, T],$$

endowed with its usual inner product. A shock on the state variable is represented by

$$\left( h^*, (h^j)_{r+1}^n, h \right) \in H,$$

where the order indicates the shift of the common spread, the specific spread and the *TSIR*, respectively. Proceeding as in Subsection 4.1, we get

$$\begin{aligned} V_q \left( h^*, (h^j)_{r+1}^n, h \right) = & \\ & \sum_{j=1}^r q_j \left( \sum_{i=1}^k \tilde{c}_{i,j} e^{-\int_0^t h(u) du} \right) + \\ & \sum_{j=r+1}^n q_j \left( \sum_{i=1}^k \tilde{c}_{i,j} e^{-\int_0^t (h^*(u) + h^j(u) + h(u)) du} \right). \end{aligned}$$

<sup>17</sup> This assumption simplifies the exposition and technicalities, but may be relaxed without altering the major results.

Due to the analogy between the expression above and (9.10), one can follow the proof of Barber and Copper (1998) to get

$$\begin{aligned} \langle \partial V_q, (h^*, (h^j)_{j=r+1}^n, h) \rangle = & - \sum_{j=1}^r q_j \left( \sum_{i=1}^k \tilde{c}_{i,j} \int_0^{t_i} h(u) \mathbf{d} \right) \\ & - \sum_{j=r+1}^n q_j \left( \sum_{i=1}^k \tilde{c}_{i,j} \int_0^{t_i} (h^*(u) + h^j(u) + h(u)) \mathbf{d} \right). \end{aligned}$$

Manipulating, we have

$$\partial V_q = - \left( \sum_{j=r+1}^n q_j \left( \sum_{t_i \geq t} \tilde{c}_{i,j} \right), \left( q_j \left( \sum_{t_i \geq t} \tilde{c}_{i,j} \right) \right)_{j=r+1}^n, \sum_{j=1}^r q_j \left( \sum_{t_i \geq t} \tilde{c}_{i,j} \right) \right) \in H, \quad (11.16)$$

and consequently,

$$\begin{aligned} \|\partial V_q\|^2 = & \left( \sum_{i=1}^k \left[ \left( \sum_{j=r+1}^n q_j \left( \sum_{l=i}^k \tilde{c}_{l,j} \right) \right)^2 (t_i - t_{i-1}) \right] \right) \\ & + \sum_{j=r+1}^n \left( \sum_{i=1}^k \left[ \left( q_j \sum_{l=i}^k \tilde{c}_{l,j} \right)^2 (t_i - t_{i-1}) \right] \right) \\ & + \left( \sum_{i=1}^k \left[ \left( \sum_{j=1}^r q_j \left( \sum_{l=i}^k \tilde{c}_{l,j} \right) \right)^2 (t_i - t_{i-1}) \right] \right). \end{aligned} \quad (11.17)$$

Thus,  $\|\partial V_q\|$  vanishes if and only if the three terms vanish, which clearly holds if and only if  $q$  replicates the zero portfolio, and  $q_j = 0, j = r + 1, r + 2, \dots, n$ . Once again, we obtain that a total immunization is not feasible unless we deal with dependent securities. So, let us analyze Criteria 2, 3 and 4. Consider the subset of  $H$  whose “ranked” elements are<sup>18</sup>

$$\begin{pmatrix} (1, (0)_{r+1}^n, 0) & (0, (0)_{r+1}^n, 1) \\ (t, (0)_{r+1}^n, 0) & (0, (0)_{r+1}^n, t) \\ (t^2, (0)_{r+1}^n, 0) & (0, (0)_{r+1}^n, t^2) \\ \dots & \dots \end{pmatrix},$$

where the order is given by the row, and therefore the column is only considered to compare two elements in the same row. Extend the subset above to a Schauder basis of  $H$  by adding

18 Once more, we are considering the basis of  $L^2[0, T]$  composed of polynomials, but the ideas of Subsection 4.2 also apply here.

$$\begin{pmatrix} (0, 1, 0, \dots, 0, 0) & (0, 0, 1, 0, \dots, 0, 0) & \dots \\ (0, t, 0, \dots, 0, 0) & (0, 0, t, 0, \dots, 0, 0) & \dots \\ (0, t^2, 0, \dots, 0, 0) & (0, 0, t^2, 0, \dots, 0, 0) & \dots \\ \dots & \dots & \dots \end{pmatrix}.$$

Apply the Gram-Schmidt process in order to get the orthonormal basis  $\{f_s\}_{s=0}^\infty$ . We have to compute the maximum number of factors making immunization feasible. Thus, bearing in mind the first component of (11.16), and proceeding as in previous subsections, we look for the highest  $R$ , such that

$$\sum_{j=r+1}^n q_j \int_0^T \left( \sum_{t_i \geq t} \tilde{c}_{i,j} \right) t^\alpha dt = 0,$$

$\alpha = 0, 1, \dots, R$ , has a solution  $q \in Q$ . Therefore,

$$\sum_{j=r+1}^n q_j \left( \sum_{i=1}^k t_i^{\alpha+1} \tilde{c}_{i,j} \right) = 0,$$

and thus,

$$\sum_{i=1}^k t_i^{\alpha+1} \left( \sum_{j=r+1}^n q_j \tilde{c}_{i,j} \right) = 0, \tag{11.18}$$

$\alpha = 0, 1, \dots, R$ . Analogously, the last component of (11.16) generates

$$\sum_{i=1}^k t_i^{\alpha+1} \left( \sum_{j=1}^n q_j \tilde{c}_{i,j} \right) = 0, \tag{11.19}$$

$\alpha = 0, 2, \dots, R$ . Hence, (11.17), (11.18) and (11.19) imply that Criterion 2 will consist in solving the quadratic mathematical programming problem<sup>19</sup>

$$\begin{aligned} \text{Min} & \left\{ \sum_{i=1}^k \left[ \left( \sum_{j=r+1}^n q_j \left( \sum_{l=i}^k \tilde{c}_{l,j} \right) \right)^2 (t_i - t_{i-1}) \right] + \sum_{j=r+1}^n \left( \sum_{i=1}^k \left[ \left( q_j \sum_{l=i}^k \tilde{c}_{l,j} \right)^2 (t_i - t_{i-1}) \right] \right) \right. \\ & \left. + \sum_{i=1}^k \left[ \left( \sum_{j=1}^n q_j \left( \sum_{l=i}^k \tilde{c}_{l,j} \right) \right)^2 (t_{r+1}^n - t_{i-1}) \right] \right\} \\ \text{s.t.} & \begin{cases} q \in Q \\ \sum_{i=1}^k t_i^{\alpha+1} \left( \sum_{j=r+1}^n q_j \tilde{c}_{i,j} \right) = 0 & \alpha = 0, 1, \dots, R \\ \sum_{i=1}^k t_i^{\alpha+1} \left( \sum_{j=1}^n q_j \tilde{c}_{i,j} \right) = 0 & \alpha = 0, 1, \dots, R \end{cases} \end{aligned}$$

Similar and straightforward computations lead to the general expressions of Criteria 3 and 4.

It is interesting to remark that, in order to respect the second restriction above, and bearing in mind that private bonds will reflect higher level of risk, the problem may often attain its optimal value

19 Note that it would be sufficient if for some  $j$  the third constraint held for  $\alpha = 0, 1, \dots, R - 1$ .

at a given point  $q$ , such that  $q_j = 0$ ,  $j = r + 1, \dots, n$ . However, the decision maker can prevent this sort of solution by drawing on the constraint  $q \in Q$ . For instance, it may impose the use of some private bonds if one claims for returns slightly bigger than those generated by public bonds.

## 5. CONCLUSION

The present paper has considered a general setting, and several properties have been established. First, we have proved the absence of immunized portfolios when infinite risk factors are considered. Second, we have provided a procedure leading to the maximum number of factors against which immunization is feasible. In this sense, since the number of factors is not previously fixed, the method seems to improve the literature considering a finite number of factors. Third, we have provided several methods to protect our strategy against the residual risk. In this sense, our analysis could improve those papers considering infinite factors, because we are hedging against more realistic shocks on the state variable. Fourth, under convexity assumptions, upper bounds for capital losses of hedged portfolios have been given. They may also provide relationships between our analysis and risk measure-linked approaches, and are still useful if convexity fails, since they yield first order approximations of the portfolio price after non favorable shocks. Fifth, we have applied our abstract method to both credit-risk-free and non-credit-risk-free portfolios of fixed income assets. Finally, further extensions have been outlined.

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